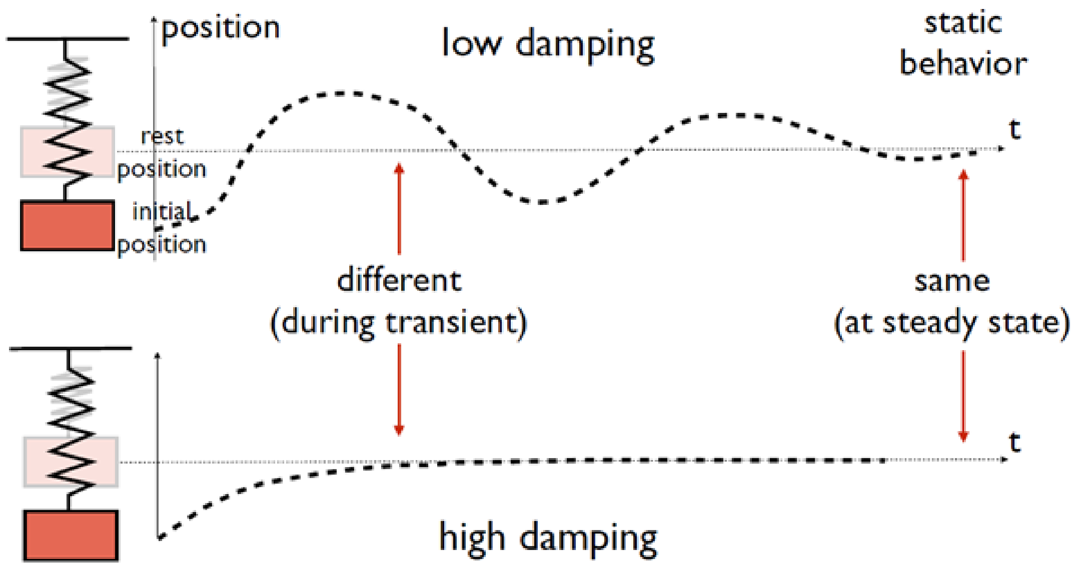
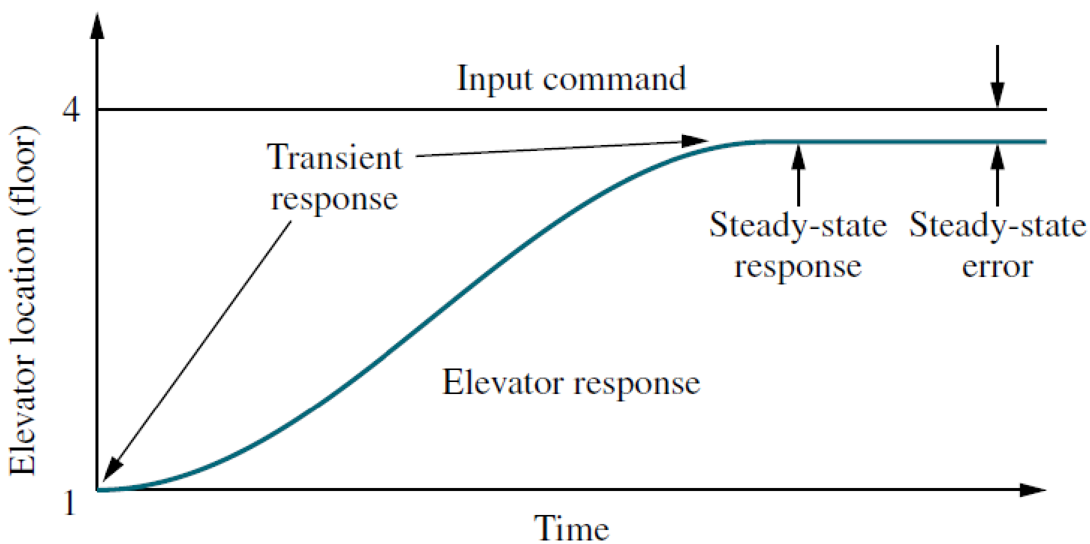
SECTION 3: STABILITY

UZM 305 – Automatic Control

Control System Design Requirements





Requirements are:

- Transient response
- Steady-state error
- Stability

Introduction

Stability

- Consider the following 2nd-order systems

$$G_1(s) = \frac{15}{(s+3)(s+5)} \quad \text{and} \quad G_2(s) = \frac{8}{s^2+4s+8}$$

- $G_1(s)$ has two real poles:

$$s_1 = -3 \quad \text{and} \quad s_2 = -5$$

- $G_2(s)$ has a complex-conjugate pair of poles:

$$s_{1,2} = -2 \pm j2$$

- The step response of each system is:

$$y_1(t) = 1.5e^{-5t} - 2.5e^{-3t} + 1$$
$$y_2(t) = -e^{-2t}[\cos(2t) + \sin(2t)] + 1$$

Stability

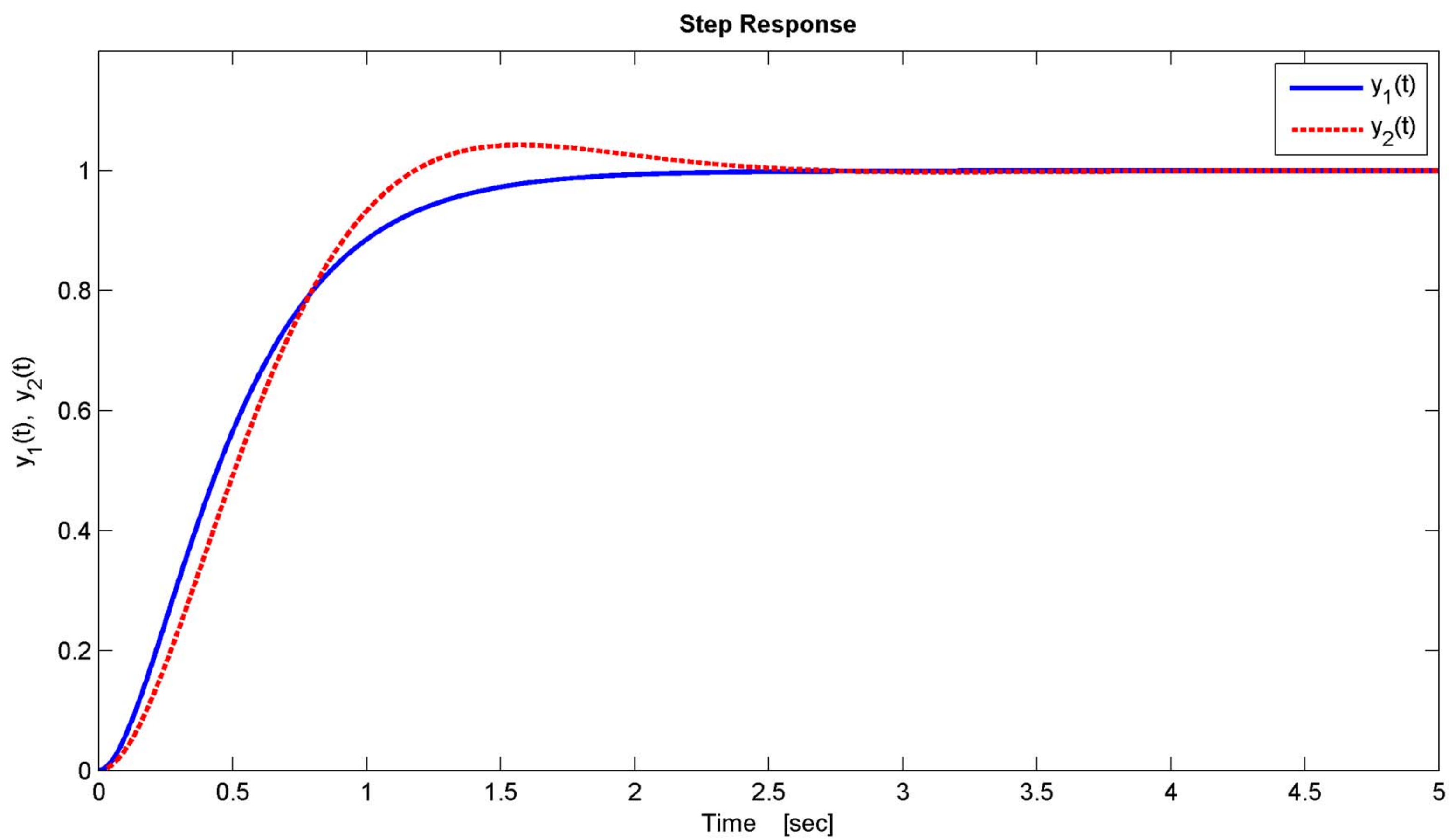
- Both step responses are a superposition of:
  - ***Natural response*** (transient)
  - ***Driven*** or ***forced response*** (steady-state)

Natural Response	Driven Response
$y_1(t) = 1.5e^{-5t} - 2.5e^{-3t}$	$+1$
$y_2(t) = -e^{-2t}[\cos(2t) + \sin(2t)]$	$+1$

- In both cases, the natural response decays to zero as $t \rightarrow \infty$

Stability

- Both step responses are characteristic of ***stable*** systems



Stability

- Now, consider the following similar-looking systems:

$$G_3(s) = \frac{15}{(s-3)(s-5)} \quad \text{and} \quad G_4(s) = \frac{8}{s^2-4s+8}$$

- $G_3(s)$ has two real poles

$$s_1 = 3 \quad \text{and} \quad s_2 = 5$$

- $G_4(s)$ has a complex-conjugate pair of poles

$$s_{1,2} = 2 \pm j2$$

- The step responses of these systems are:

$$y_3(t) = 1.5e^{5t} - 2.5e^{3t} + 1$$

$$y_4(t) = -e^{2t}[\cos(2t) + \sin(2t)] + 1$$

Stability

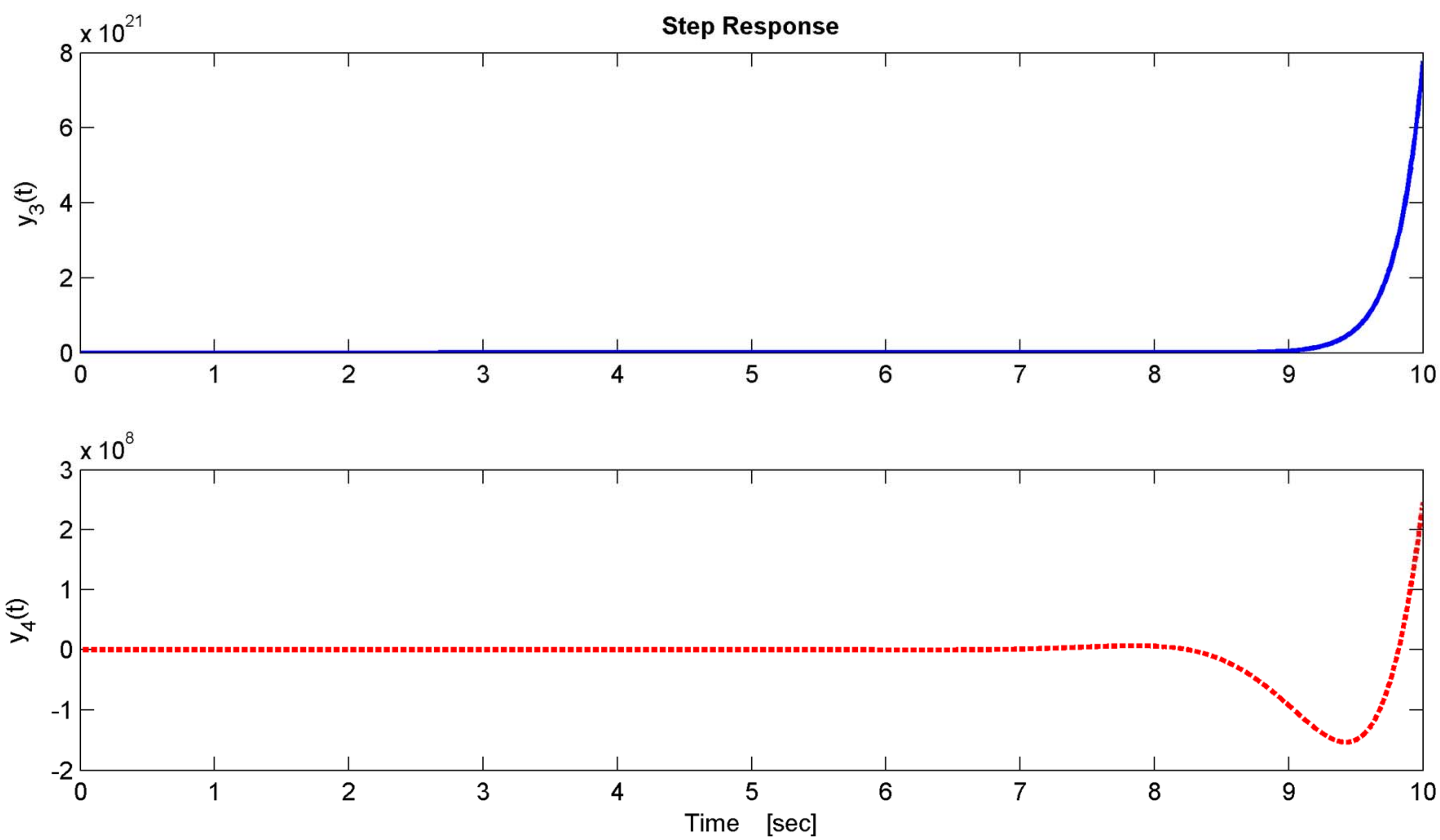
- Again, step responses consist of a natural response component and a driven component

Natural Response	Driven Response
$y_1(t) = 1.5e^{5t} - 2.5e^{3t}$	$+1$
$y_2(t) = -e^{2t}[\cos(2t) + \sin(2t)]$	$+1$

- Now, as $t \to \infty$, the natural responses do not decay to zero
  - They blow up – why?
  - ***Exponential terms are positive***

Stability

- Step responses characteristic of ***unstable*** systems



Stability

- Why are the exponential terms positive?
  - Determined by the system poles
- For the over-damped system, the poles are

$$s_1 = \sigma_1 \quad \text{and} \quad s_2 = \sigma_2$$

- And, the step response is

$$y(t) = r_1 e^{\sigma_1 t} + r_2 e^{\sigma_2 t} + r_3$$

- For the under-damped system, the poles are

$$s_{1,2} = \sigma \pm j\omega_d$$

- The step response is

$$y(t) = r_1 e^{\sigma t} \cos(\omega_d t) + r_2 e^{\sigma t} \sin(\omega_d t) + r_3$$

Stability and System Poles

- Sign of the exponentials determined by σ, the ***real part of the system poles***
- If $\sigma < 0$
  - Pole is in the ***left half-plane*** (LHP)
  - Natural response $\rightarrow 0$ as $t \rightarrow \infty$
  - System is ***stable***
- If $\sigma > 0$
  - Pole is in the ***right half-plane*** (RHP)
  - Natural response $\rightarrow \infty$ as $t \rightarrow \infty$
  - System is ***unstable***

Purely-Imaginary Poles

- ***LHP poles*** correspond to ***stable*** systems
- ***RHP poles*** correspond to ***unstable*** systems
- It seems that the imaginary axis is the boundary for stability
- What if poles are on the imaginary axis?
- Consider the following system

$$G_5(s) = \frac{4}{s^2 + 4}$$

- Two purely-imaginary poles

$$s_{1,2} = \pm j2$$

Marginal Stability

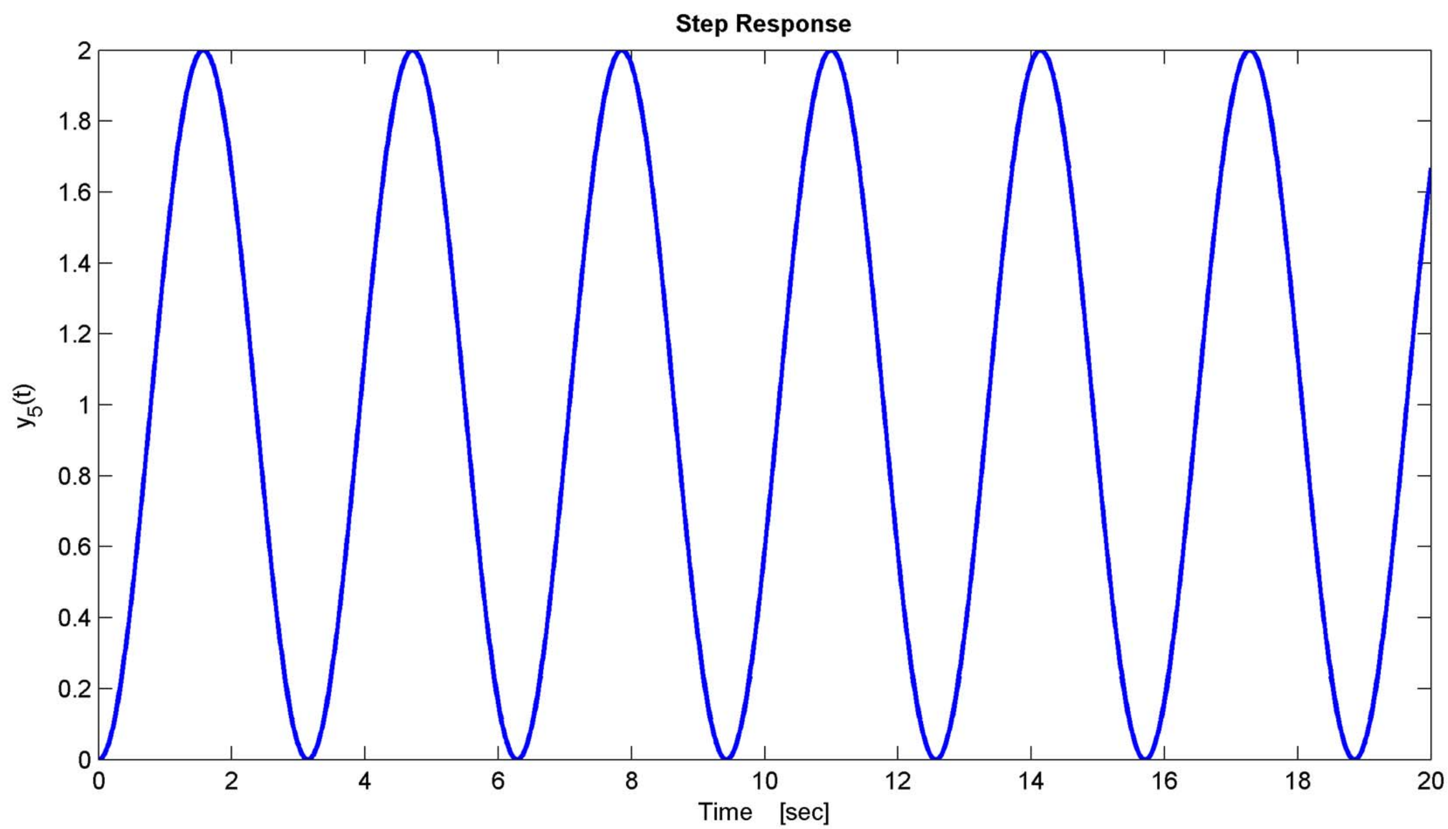
- Step response for this ***undamped system*** is

Natural Response	Driven Response
$y_5(t) = -\cos(2t)$	$+1$

- Natural response neither decays to zero, nor grows without bound
  - Oscillates indefinitely
  - System is ***marginally stable***

Marginal Stability

- Step response is characteristic of a ***marginally-stable*** system



Repeated Imaginary Poles

- We'll look at one more interesting case before presenting a formal definition for stability
- Consider the following system

$$G_6(s) = \frac{16}{s^4 + 8s^2 + 16} = \frac{16}{(s^2+4)^2}$$

- Repeated poles on the imaginary axis

$$s_{1,2} = \pm j2 \quad \text{and} \quad s_{3,4} = \pm j2$$

- The step response for this system is

Natural Response	Driven Response
$y_6(t) = -\cos(2t) - t \cdot \sin(2t)$	$+1$

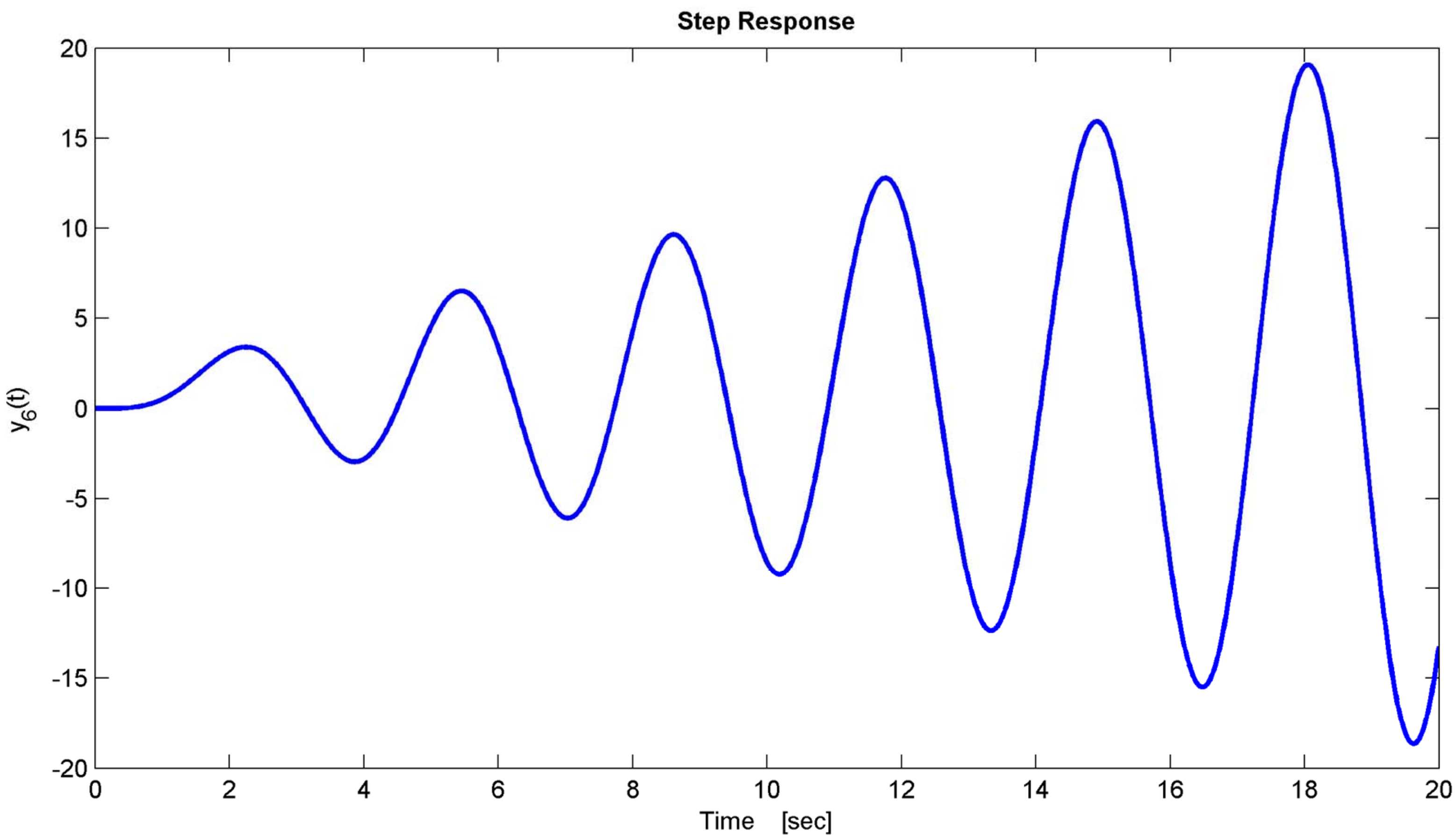
Repeated Imaginary Poles

$$y_6(t) = -\cos(2t) - t \cdot \sin(2t) + 1$$

- Multiplying time factor causes the natural response to grow without bound
  - An ***unstable system***
  - Results from repeated poles
- ***Multiple identical poles on the imaginary axis implies an unstable system***

Repeated Imaginary Poles

- Step response shows that the system is unstable



Definitions of Stability

Definitions of Stability – Natural Response

- We know that system response is the sum of a natural response and a driven response
- Can define the categories of stability based on the ***natural response***:
- ***Stable***
  - A system is stable if its natural response $\rightarrow 0$ as $t \rightarrow \infty$
- ***Unstable***
  - A system is unstable if its natural response $\rightarrow \infty$ as $t \rightarrow \infty$
- ***Marginally Stable***
  - A system is marginally stable if its natural response neither decays nor grows, but remains constant or oscillates

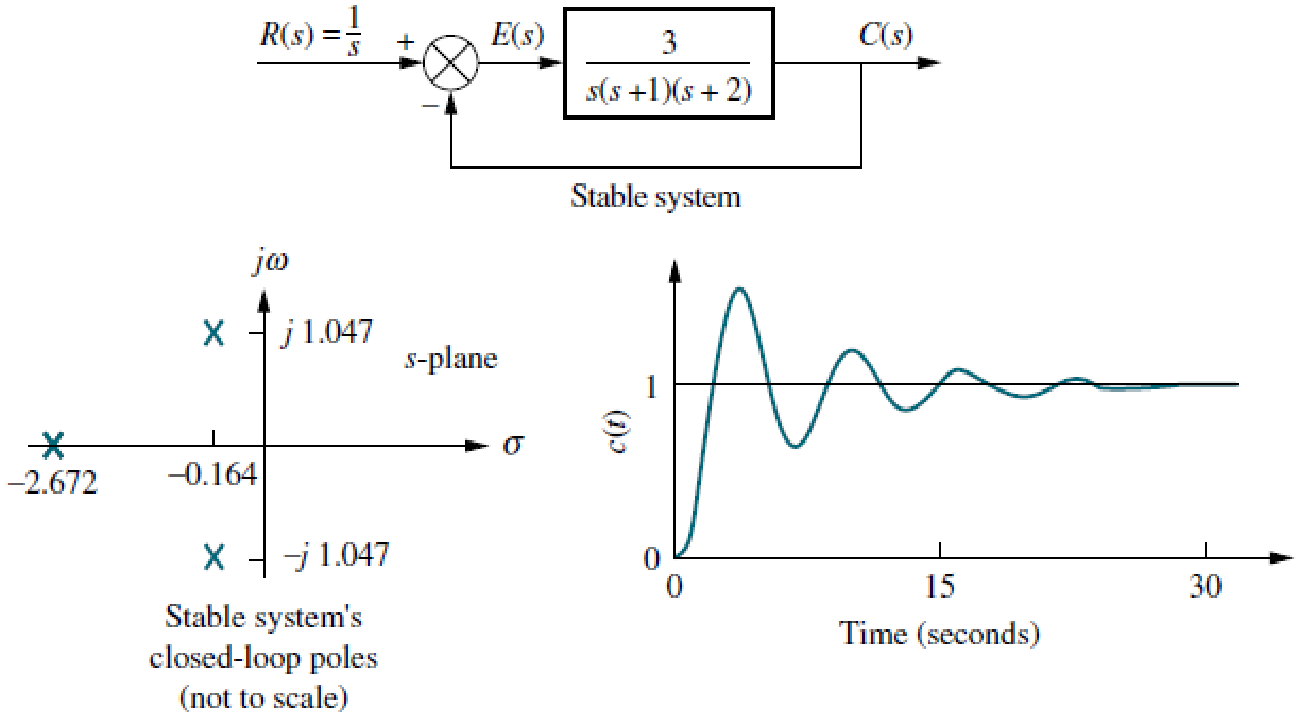
BIBO Stability

- Alternatively, we can define stability based on the total response
- ***Bounded-input, bounded-output*** (***BIBO***) ***stability***
- ***Stable***
  - A system is stable if *every* bounded input yields a bounded output
- ***Unstable***
  - A system is unstable if *any* bounded input yields an unbounded output

Closed-Loop Poles and Stability

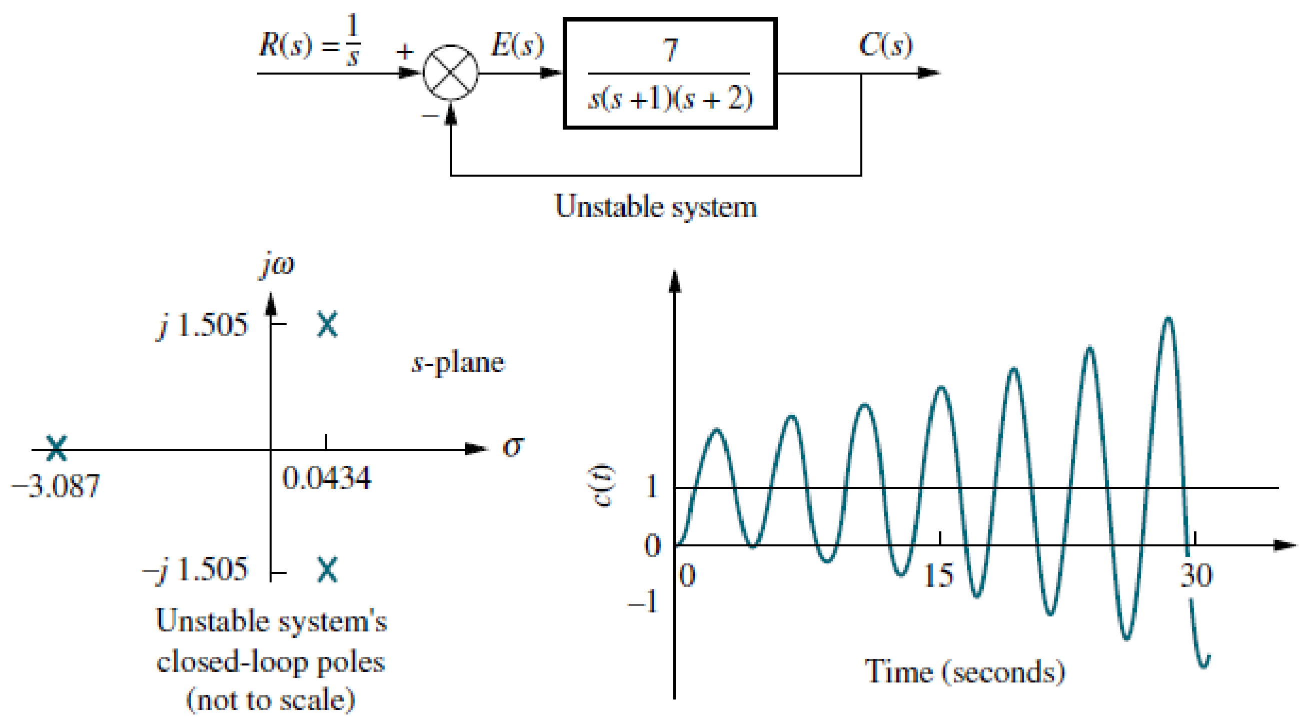
- ***Stable***
  - A stable system has all of its closed-loop poles in the left-half plane
- ***Unstable***
  - An unstable system has at least one pole in the right half-plane and/or repeated poles on the imaginary axis
- ***Marginally Stable***
  - A marginally-stable system has non-repeated poles on the imaginary axis and (possibly) poles in the left half-plane

Determining System Stability



$R(s) = \frac{1}{s}$
+
−
$E(s)$
$\frac{3}{s(s+1)(s+2)}$
$C(s)$
Stable system
$j\omega$
j 1.047
s-plane
σ
−2.672
−0.164
$-j$ 1.047
Stable system's
closed-loop poles
(not to scale)
$c(t)$
1
0
0
15
30
Time (seconds)

(*a*)



$R(s) = \frac{1}{s}$
+
−
$E(s)$
$\frac{7}{s(s+1)(s+2)}$
$C(s)$
Unstable system
$j\omega$
j 1.505
s-plane
σ
−3.087
0.0434
$-j$ 1.505
Unstable system's
closed-loop poles
(not to scale)
$c(t)$
1
0
−1
0
15
30
Time (seconds)

(*b*)

Determining Stability

- Stability determined by pole locations
  - Poles determined by the characteristic polynomial, $\Delta(s)$
- Factoring the characteristic polynomial will always tell us if a system is stable or not
  - Easily done with a computer or calculator
- If you have an unknown parameter in the denominator of a transfer function, it is difficult to determine via a calculator the range of this parameter to yield stability
  - Form of $\Delta(s)$ may indicate RHP poles directly, or
  - Routh-Hurwitz Criterion

Stability from $\Delta(s)$ Coefficients

- A stable system has all poles in the LHP

$$T(s) = \frac{Num(s)}{(s+a_1)(s+a_2)\cdots(s+a_n)}$$

  - Poles: $p_i = -a_i$
  - For all LHP poles, $a_i > 0, \quad \forall i$
  - Result is that all coefficients of $\Delta(s)$ are ***positive***

- If any coefficient of $\Delta(s)$ is ***negative***, there is at least one RHP pole, and the system is ***unstable***
- If any coefficient of $\Delta(s)$ is ***zero***, the system is ***unstable*** or, at best, ***marginally stable***
- If all coefficients of $\Delta(s)$ are ***positive***, the system may be ***stable*** or may be ***unstable***

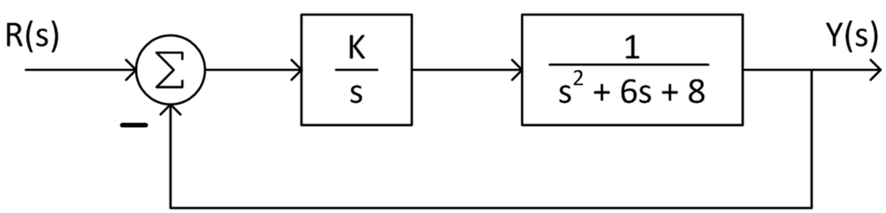
Routh-Hurwitz Criterion

- Need a method to detect RHP poles if all coefficients of $\Delta(s)$ are positive:
  - ***Routh-Hurwitz criterion***

- General procedure:
  1. Generate a ***Routh table*** using the characteristic polynomial of the closed-loop system
  2. Apply the ***Routh-Hurwitz criterion*** to interpret the table and determine the *number* (not locations) of RHP poles

Routh-Hurwitz – Utility

- Routh-Hurwitz was very useful for determining stability in the days before computers
  - Factoring polynomials by hand is difficult
- Still useful for ***design***, e.g.:



$$T(s) = \frac{K}{s^3 + 6s^2 + 8s + K}$$

- Stable for some range of gain, K, but unstable beyond that range
- Routh-Hurwitz allows us to determine that range

Routh Table

- Consider a 4th-order closed-loop transfer function:

$$T(s) = \frac{Num(s)}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

- Routh table has one row for each power of s in $\Delta(s)$
  - First row contains coefficients of even powers of s (odd if the order of $\Delta(s)$ is odd)
  - Second row contains coefficients of odd (even) powers of s
  - Fill in zeros if needed – if even order

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

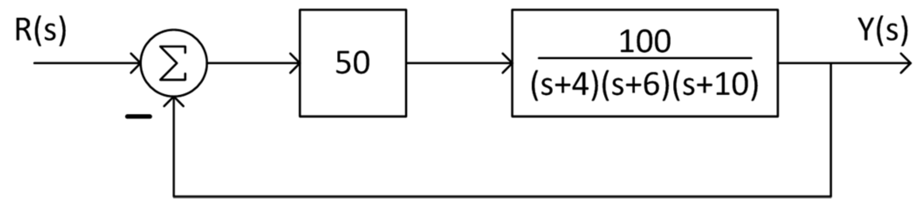
Routh Table

- Remaining table entries calculated using entries from two preceding rows as follows:

$$
\begin{array}{c|ccc}
s^4 & a_4 & a_2 & a_0 \\
s^3 & a_3 & a_1 & 0 \\
s^2 & -\dfrac{\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1 & -\dfrac{\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2 & -\dfrac{\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_3 = 0 \\
s^1 & -\dfrac{\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1 & -\dfrac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = c_2 = 0 & -\dfrac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = c_3 = 0 \\
s^0 & -\dfrac{\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1 & -\dfrac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_2 = 0 & -\dfrac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_3 = 0
\end{array}
$$

Routh Table – Example

- Consider the following feedback system



- The closed-loop transfer function is

$$T(s) = \frac{5000}{s^3 + 20s^2 + 124s + 5240}$$

- The first two rows of the Routh table are

s^3	1	124
s^2	~~20~~ 1	~~5240~~ 262

- Note that we can simplify by scaling an entire row by any factor

Routh Table – Example

- Calculate the remaining table entries:

s^3	1	124
s^2	$\not{20}\ 1$	$\not{5240}\ 262$
s^1	$-\dfrac{\begin{vmatrix}1 & 124\\ 1 & 262\end{vmatrix}}{1} = -138$	$-\dfrac{\begin{vmatrix}1 & 0\\ 1 & 0\end{vmatrix}}{1} = 0$
s^0	$-\dfrac{\begin{vmatrix}1 & 262\\ -138 & 0\end{vmatrix}}{-138} = 262$	$-\dfrac{\begin{vmatrix}1 & 0\\ -138 & 0\end{vmatrix}}{-138} = 0$

- How do we interpret this table?
  - ***Routh-Hurwitz criterion***

Routh-Hurwitz Criterion

- ***Routh-Hurwitz Criterion***
  - *The number of poles in the RHP is equal to the number of sign changes in the first column of the Routh table*

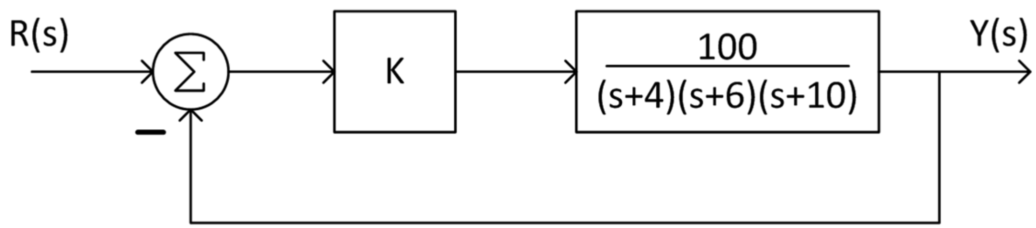
- Apply this criterion to our example:

s^3	1	124
s^2	1	262
s^1	-138	0
s^0	262	0

- Two sign changes in the first column indicate ***two RHP poles*** → system is ***unstable***

Routh-Hurwitz – Stability Requirements

- Consider the same system, where controller gain is left as a parameter



- Closed-loop transfer function:

$$T(s) = \frac{100K}{s^3 + 20s^2 + 124s + 240 + 100K}$$

- Plant itself is stable
  - Presumably there is some range of gain, K, for which the closed-loop system is also stable
  - Use ***Routh-Hurwitz*** to determine this range

Routh-Hurwitz – Stability Requirements

$$T(s) = \frac{100K}{s^3 + 20s^2 + 124s + 240 + 100K}$$

- Create the Routh table

$$\begin{array}{c|cc}
s^3 & 1 & 124 \\
s^2 & \cancel{20}\ 1 & \cancel{240 + 100K}\ 12 + 5K \\
s^1 & -\dfrac{\begin{vmatrix} 1 & 124 \\ 1 & 12+5K \end{vmatrix}}{1} = 112 - 5K & -\dfrac{\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0 \\
s^0 & -\dfrac{\begin{vmatrix} 1 & 12+5K \\ 112-5K & 0 \end{vmatrix}}{112-5K} = 12 + 5K & -\dfrac{\begin{vmatrix} 1 & 0 \\ 112\text{-}5K & 0 \end{vmatrix}}{112-5K} = 0
\end{array}$$

Routh-Hurwitz – Stability Requirements

s^3	1	124
s^2	1	$12+5K$
s^1	$112-5K$	0
s^0	$12+5K$	0

- Since $K > 0$, only the third element in the first column can be negative
- ***Stable*** for

$$112 - 5K > 0$$

$$K < 22.4$$

- ***Unstable*** (two RHP poles) for

$$112 - 5K < 0$$
$$K > 22.4$$

Routh Table – Special Cases

- Two special cases can arise when creating a Routh table:
  1. ***A zero in only the first column of a row***
     - Divide-by-zero problem when forming the next row
  2. ***An entire row of zeros***
     - Indicates the presence of pairs of poles that are mirrored about the imaginary axis
- We'll next look at methods for dealing with each of these scenarios

Routh Table – Zero in the First Column

- If a zero appears in the first column
  1. Replace the zero with $\pm\epsilon$
  2. Complete the Routh table as usual
  3. $\epsilon \to 0$, from either the positive or the negative side
  4. Evaluate the sign of the first-column entries

- For example:

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

- First two rows in the Routh table:

s^5	1	3	5
s^4	2	6	3

First-Column Zero – Example

$$
\begin{array}{c|ccc}
s^5 & 1 & 3 & 5 \\
s^4 & 2 & 6 & 3 \\
s^3 & -\dfrac{\begin{vmatrix}1 & 3\\ 2 & 6\end{vmatrix}}{2} = \not{0}\ \epsilon & -\dfrac{\begin{vmatrix}1 & 5\\ 2 & 3\end{vmatrix}}{2} = 7/2 & -\dfrac{\begin{vmatrix}1 & 0\\ 2 & 0\end{vmatrix}}{2} = 0
\end{array}
$$

- Replace the first-column zero with ϵ and proceed as usual

$$
\begin{array}{c|ccc}
s^2 & -\dfrac{\begin{vmatrix}1 & 6\\ \epsilon & 7\end{vmatrix}}{\epsilon} = \dfrac{6\epsilon - 7}{\epsilon} & -\dfrac{\begin{vmatrix}2 & 3\\ \epsilon & 0\end{vmatrix}}{\epsilon} = 3 & -\dfrac{\begin{vmatrix}2 & 0\\ \epsilon & 0\end{vmatrix}}{\epsilon} = 0 \\
s^1 & -\dfrac{\begin{vmatrix}\epsilon & 3\\ \frac{6\epsilon - 7}{\epsilon} & 3\end{vmatrix}}{\frac{6\epsilon - 7}{\epsilon}} = \dfrac{42\epsilon\text{-}\ 49\text{-}6\epsilon^2}{12\epsilon - 14} & -\dfrac{\begin{vmatrix}\epsilon & 0\\ \frac{6\epsilon - 7}{\epsilon} & 0\end{vmatrix}}{\frac{6\epsilon - 7}{\epsilon}} = 0 & -\dfrac{\begin{vmatrix}\epsilon & 0\\ \frac{6\epsilon - 7}{\epsilon} & 0\end{vmatrix}}{\frac{6\epsilon - 7}{\epsilon}} = 0
\end{array}
$$

- Continuing on the next page …

First-Column Zero – Example

s^5	1	2	6
s^4	1	2	3
s^3	ϵ	3	0
s^2	$\frac{6\epsilon - 7}{\epsilon}$	3	0
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0
s^0	$-\frac{\begin{vmatrix} \frac{6\epsilon - 7}{\epsilon} & 3 \\ \frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14} & 0 \end{vmatrix}}{\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}} = 3$	$-\frac{\begin{vmatrix} \frac{6\epsilon - 7}{\epsilon} & 0 \\ \frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14} & 0 \end{vmatrix}}{\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}} = 0$	$-\frac{\begin{vmatrix} \frac{6\epsilon - 7}{\epsilon} & 0 \\ \frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14} & 0 \end{vmatrix}}{\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}} = 0$

- Next, take the $\epsilon \rightarrow 0$

First-Column Zero – Example

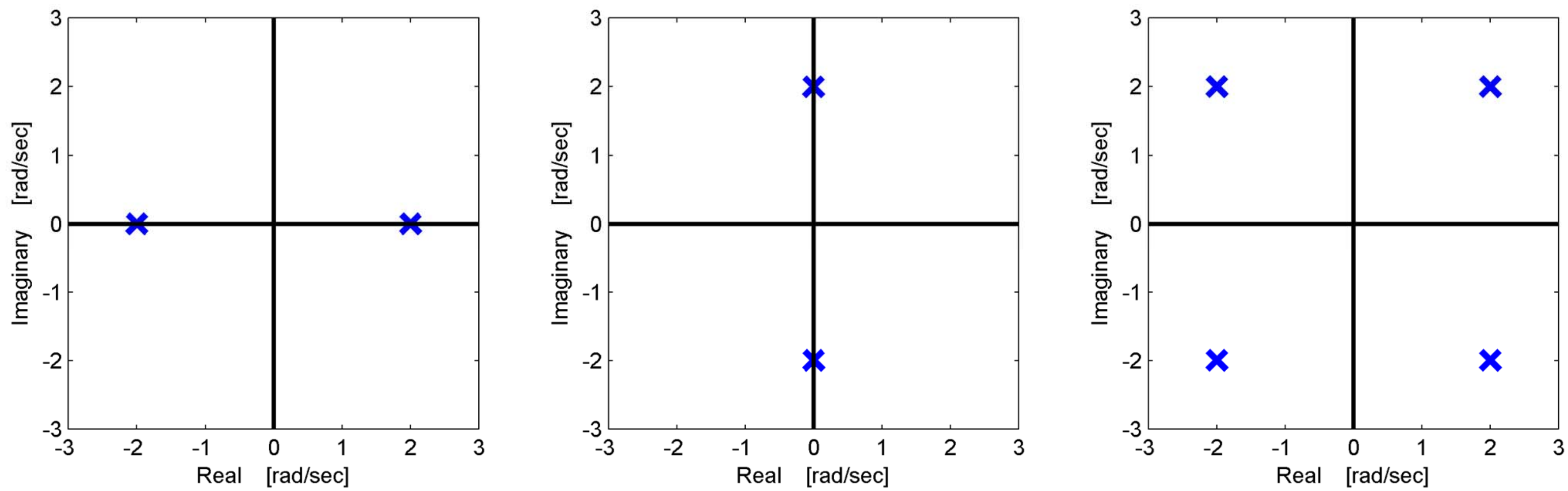
- Approach $\epsilon \to 0$ and looking at the first column:

Label	First column	$\epsilon = +$	$\epsilon = -$
s^5	1	+	+
s^4	2	+	+
s^3	~~0~~ ϵ	+	−
s^2	$\frac{6\epsilon - 7}{\epsilon}$	−	+
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	+	+
s^0	3	+	+

- Two sign changes
  - Two RHP poles
  - System is ***unstable***

Routh Table – Row of Zeros

- A whole row of zeros indicates the presence of pairs of poles that are mirrored about the imaginary axis:



- At best, the system is ***marginally stable***
- Use a Routh table to determine if it is ***unstable***

Routh Table – Row of Zeros

- If an entire row of zeros appears in a Routh table
  1. Create an ***auxiliary polynomial*** from the row above the row of zeros, skipping every other power of s
  2. Differentiate the auxiliary polynomial w.r.t. s
  3. Replace the zero row with the coefficients of the resulting polynomial
  4. Complete the Routh table as usual
  5. Evaluate the sign of the first-column entries

Row of Zeros – Example

- Consider the following system

$$T(s) = \frac{1}{s^5 + 5s^4 + 11s^3 + 23s^2 + 28s + 12}$$

- The first few rows of the Routh table:

s^5	1	11	28
s^4	5	23	12
s^3	$-\frac{\begin{vmatrix}1 & 11\\5 & 23\end{vmatrix}}{5} = \cancel{6.4}\ 1$	$-\frac{\begin{vmatrix}1 & 28\\5 & 12\end{vmatrix}}{5} = \cancel{25.6}\ 4$	$-\frac{\begin{vmatrix}1 & 0\\5 & 0\end{vmatrix}}{5} = 0$
s^2	$-\frac{\begin{vmatrix}5 & 23\\1 & 4\end{vmatrix}}{1} = \cancel{3}\ 1$	$-\frac{\begin{vmatrix}5 & 12\\1 & 0\end{vmatrix}}{1} = \cancel{12}\ 4$	$-\frac{\begin{vmatrix}5 & 0\\1 & 0\end{vmatrix}}{1} = 0$

- Continuing on the next page …

Row of Zeros – Example

s^5	1	11	28
s^4	5	23	12
s^3	1	4	0
s^2	1	4	0
s^1	$-\frac{\begin{vmatrix}1 & 4\\1 & 4\end{vmatrix}}{1}=0$	$-\frac{\begin{vmatrix}1 & 0\\1 & 0\end{vmatrix}}{1}=0$	$-\frac{\begin{vmatrix}1 & 0\\1 & 0\end{vmatrix}}{1}=0$

- A row of zeros has appeared
  - Create an auxiliary polynomial from the s^2 row

$$P(s) = s^2 + 4$$

  - Differentiate

$$\frac{dP}{ds} = 2s$$

  - Replace the s^1 row with the dP/ds coefficients

Row of Zeros – Example

$$\frac{dP}{ds} = 2s$$

- Replacing the s^1 row with the coefficients of dP/ds

s^5	1	11	28
s^4	5	23	12
s^3	1	4	0
s^2	1	4	0
s^1	~~0~~ 2	0	0
s^0	$-\frac{\begin{vmatrix} 1 & 4 \\ 2 & 0 \end{vmatrix}}{2} = 4$	$-\frac{\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix}}{2} = 0$	$-\frac{\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix}}{2} = 0$

- No sign changes, so RHP poles, *but*
  - Row of zeros indicates that system is ***marginally stable***

Stability Evaluation – Summary

- If coefficients of $\Delta(s)$ have different signs
  - System is unstable
- If some coefficients of $\Delta(s)$ are zero
  - System is, at best, marginally stable
- If all $\Delta(s)$ coefficients have the same sign
  - System may be stable or unstable
  - Generate a Routh table and apply Routh-Hurwitz criterion
  - Replace any zero first-column entries with and let take the limit as $\epsilon \to 0$
  - Replace a row of zeros with coefficients from the derivative of the auxiliary polynomial
    - If no RHP poles are detected, the system is marginally stable
  - **System is stable if all of the poles are only in the left half-plane**