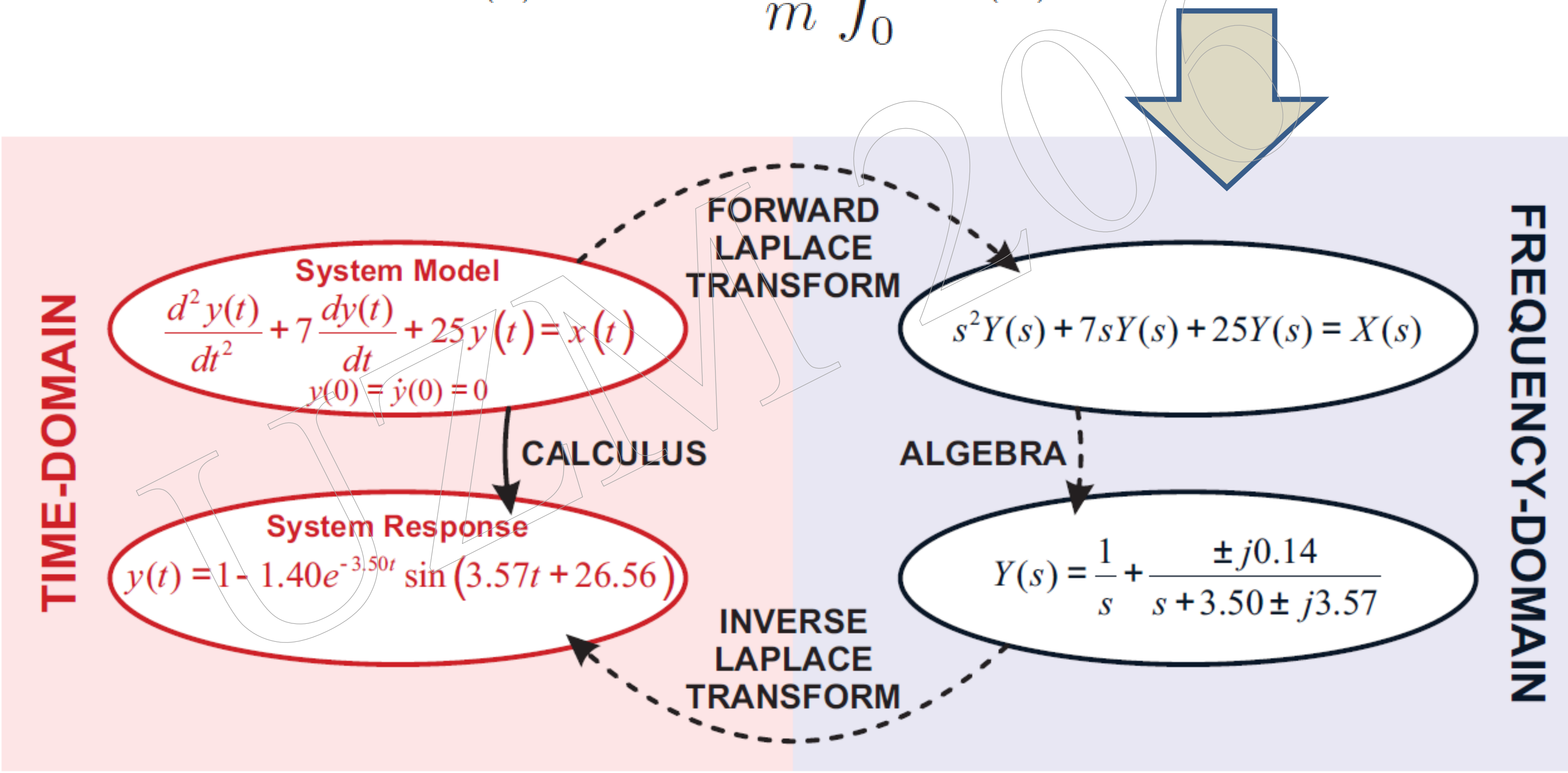
UZM 206
Sistem Dinamiği
System Dynamics

Bölüm 3: Frekans Alanında Modelleme

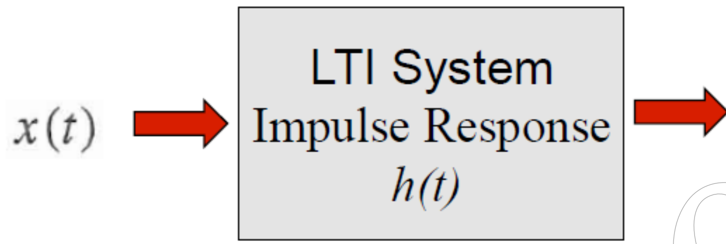
Dr. Harun ÇELİK
Uzay Mühendisliği Bölümü
haruncelik@erciyes.edu.tr

Doğrusal Sistem Yanıtı

$$v(t) = v_0 + \frac{1}{m}\int_0^t F(\tau)d\tau$$

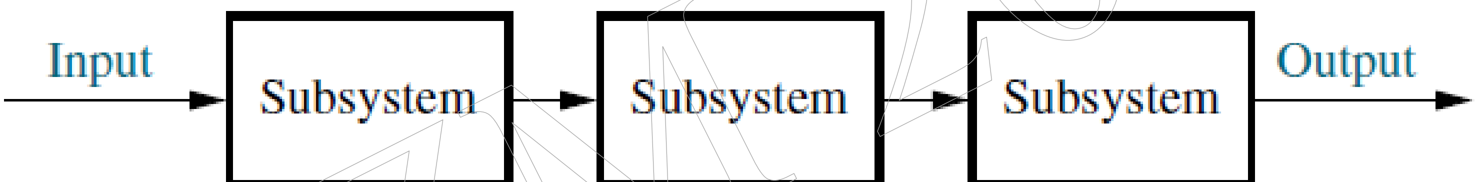


Doğrusal Sistem Yanıtı



$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$$



- Zaman ekseninde diferansiyel denklemle ifade edilen bir sistemin blok şemalarla modellenmesi zordur.
- Laplace dönüşümü, sistemi zaman alanından frekans alanına aktarır.
- Laplace dönüşümü ile giriş, çıkış ve sistem ayrı olarak temsil edilebilir. Aralarındaki ilişkiler de basit cebirsel olur.

Laplace Dönüşümü

The Laplace transform is defined as

$$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$$

where $s = \sigma + j\omega$, a complex variable. Thus, knowing $f(t)$ and that the integral in Eq. exists, we can find a function, $F(s)$, that is called the *Laplace transform* of $f(t)$.

The inverse Laplace transform, which allows us to find $f(t)$ given $F(s)$, is

$$\mathscr{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}\,ds = f(t)u(t)$$

where

$$\begin{aligned} u(t) &= 1 \qquad t > 0 \\ &= 0 \qquad t < 0 \end{aligned}$$

is the unit step function. Multiplication of $f(t)$ by $u(t)$ yields a time function that is zero for $t < 0$.

Laplace Dönüşümleri

Laplace transform table

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Example

Find the Laplace transform of $f(t) = Ae^{-at}u(t)$.

Since the time function does not contain an impulse function, we can replace the lower limit of Eq. with 0. Hence,

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty Ae^{-at}e^{-st}\,dt = A\int_0^\infty e^{-(s+a)t}\,dt$$

$$= -\frac{A}{s+a}e^{-(s+a)t}\Big|_{t=0}^{\infty} = \frac{A}{s+a}$$

Laplace Dönüşüm Özellikleri

Laplace transform theorems

Theorem		Name
$\mathscr{L}[f(t)] = F(s)$	$= \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
$\mathscr{L}[kf(t)]$	$= kF(s)$	Linearity theorem
$\mathscr{L}[f_1(t) + f_2(t)]$	$= F_1(s) + F_2(s)$	Linearity theorem
$\mathscr{L}[e^{-at}f(t)]$	$= F(s+a)$	Frequency shift theorem
$\mathscr{L}[f(t-T)]$	$= e^{-sT}F(s)$	Time shift theorem
$\mathscr{L}[f(at)]$	$= \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
$\mathscr{L}\left[\frac{df}{dt}\right]$	$= sF(s) - f(0-)$	Differentiation theorem
$\mathscr{L}\left[\frac{d^2f}{dt^2}\right]$	$= s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
$\mathscr{L}\left[\frac{d^nf}{dt^n}\right]$	$= s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
$\mathscr{L}\left[\int_{0-}^{t} f(\tau)d\tau\right]$	$= \frac{F(s)}{s}$	Integration theorem
$f(\infty)$	$= \lim_{s\to 0} sF(s)$	Final value theorem[1]
$f(0+)$	$= \lim_{s\to \infty} sF(s)$	Initial value theorem[2]

[1]For this theorem to yield correct finite results, all roots of the denominator of $F(s)$ must have negative real parts, and no more than one can be at the origin.

[2]For this theorem to be valid, $f(t)$ must be continuous or have a step discontinuity at $t = 0$ (that is, no impulses or their derivatives at $t = 0$).

Laplace Dönüşümü - Örnek

Use the Laplace transform to solve the differential equation

$$\frac{d^2v(t)}{dt^2} + 6\frac{dv(t)}{dt} + 8v(t) = 2u(t)$$

subject to $v(0) = 1, v'(0) = -2$.

Solution:
We take the Laplace transform of each term in the given differential equation and obtain

$$[s^2V(s) - sv(0) - v'(0)] + 6[sV(s) - v(0)] + 8V(s) = \frac{2}{s}$$

Substituting $v(0) = 1, v'(0) = -2$,

$$s^2V(s) - s + 2 + 6sV(s) - 6 + 8V(s) = \frac{2}{s}$$

or

$$(s^2 + 6s + 8)V(s) = s + 4 + \frac{2}{s} = \frac{s^2 + 4s + 2}{s}$$

Hence,

Basit Kesirlere Ayırma

$$V(s) = \frac{s^2 + 4s + 2}{s(s + 2)(s + 4)} = \frac{A}{s} + \frac{B}{s + 2} + \frac{C}{s + 4}$$

where

$$A = sV(s)\,|_{s=0} = \frac{s^2 + 4s + 2}{(s + 2)(s + 4)}\bigg|_{s=0} = \frac{2}{(2)(4)} = \frac{1}{4}$$

$$B = (s + 2)V(s)\,|_{s=-2} = \frac{s^2 + 4s + 2}{s(s + 4)}\bigg|_{s=-2} = \frac{-2}{(-2)(2)} = \frac{1}{2}$$

$$C = (s + 4)V(s)\,|_{s=-4} = \frac{s^2 + 4s + 2}{s(s + 2)}\bigg|_{s=-4} = \frac{2}{(-4)(-2)} = \frac{1}{4}$$

Hence,

$$V(s) = \frac{\frac{1}{4}}{s} + \frac{\frac{1}{2}}{s + 2} + \frac{\frac{1}{4}}{s + 4}$$

By the inverse Laplace transform,

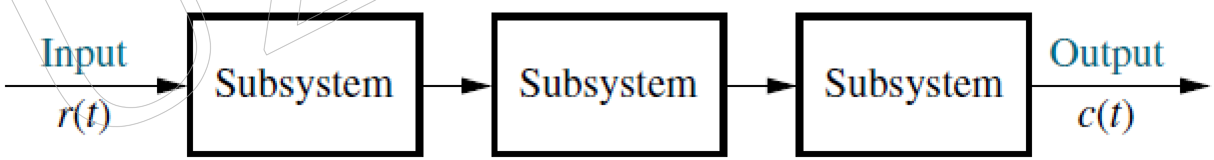
$$v(t) = \frac{1}{4}(1 + 2e^{-2t} + e^{-4t})u(t)$$

Transfer Fonksiyonu

- Zaman ekseninde diferansiyel denklemlerle ifade edilen ilişkiler frekans eksenine geçince transfer fonksiyonlarıyla ifade edilir.
- Bu fonksiyon, diferansiyel denklemden **farklı olarak** giriş, sistem ve çıkışın üç ayrı ve farklı parçaya ayrılmasına olanak sağlar.
- *n* inci mertebeden, doğrusal ve zamandan bağımsız diferansiyel denklem:

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

- formunda ve ayrıştırılarak çözülmesi zor iken, aşağıdaki gibi alt bileşenlere ayrılması transfer fonksiyonlarında mümkündür



Transfer Fonksiyonu

$$a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \cdots + a_0 C(s) + \text{initial condition terms involving } c(t)$$

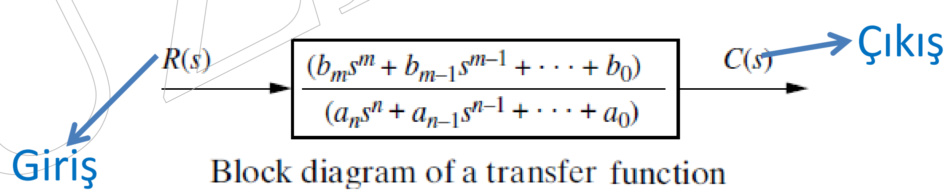
$$= b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \cdots + b_0 R(s) + \text{initial condition terms involving } r(t)$$

Equation is a purely algebraic expression. If we assume that *all initial conditions are zero*, Eq. reduces to

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0) C(s) = (b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0) R(s)$$

Now form the ratio of the output transform, $C(s)$, divided by the input transform, $R(s)$:

$$\boxed{\frac{C(s)}{R(s)} = G(s) = \frac{(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0)}}$$



Block diagram of a transfer function

$$\boxed{C(s) = R(s)G(s)}$$

Transfer Fonksiyonu

Transfer Fonksiyonuyla Çözme - Örnek

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

Laplace dönüşümü:

$$sC(s) + 2C(s) = R(s)$$

Transfer fonksiyonu:

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2}$$

Giriş $r(t) = u(t)$, şeklinde

birim basamak fonksiyonu olursa

Laplace dönüşümü: $R(s) = 1/s,$

Çıkış:

$$C(s) = R(s)G(s) = \frac{1}{s(s+2)}$$

$$C(s) = \frac{1/2}{s} - \frac{1/2}{s+2}$$

Frekans eksenindeki bu çıkışın Ters Laplace dönümü alınırsa Zaman ekseninde çıkış:

$$c(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$$

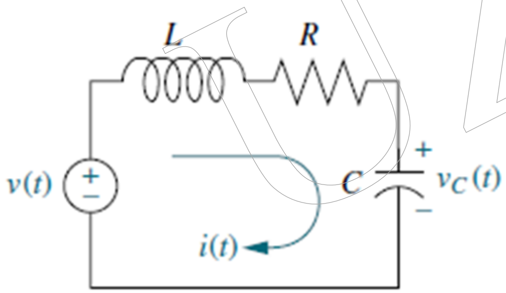
Elektrik Devresi Transfer Fonksiyonları

Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ – V (volts), $i(t)$ – A (amps), $q(t)$ – Q (coulombs), C – F (farads), R – Ω (ohms), G – Ω (mhos), L – H (henries).

Alıştırma



RLC network

Yandaki devre için kapasitör gerilimi $Vc(s)$ ile giriş gerilimi $V(s)$ arasındaki ilişkiyi ifade eden transfer fonksiyonunu bulunuz.

Çözüm

$$L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\int_0^t i(\tau)d\tau=v(t)$$

using $i(t)=dq(t)/dt$ yields

$$L\frac{d^2q(t)}{dt^2}+R\frac{dq(t)}{dt}+\frac{1}{C}q(t)=v(t)$$

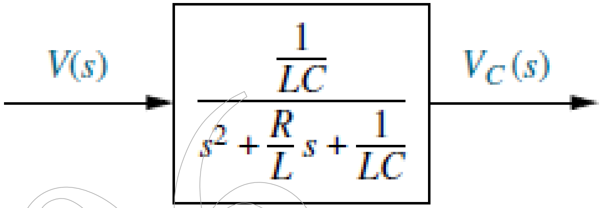
relationship for a capacitor

$$q(t)=Cv_C(t)$$

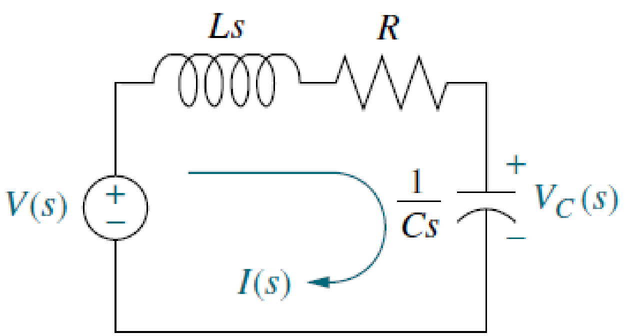
$$LC\frac{d^2v_C(t)}{dt^2}+RC\frac{dv_C(t)}{dt}+v_C(t)=v(t)$$

$$(LCs^2+RCs+1)V_C(s)=V(s)$$

Transfer Fonksiyonu: $$\frac{V_C(s)}{V(s)}=\frac{1/LC}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$



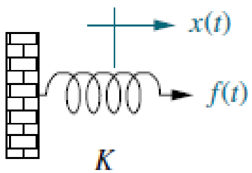
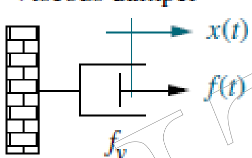
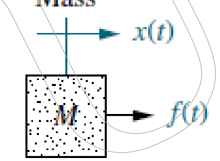
Block diagram of series RLC electrical network



Laplace-transformed network

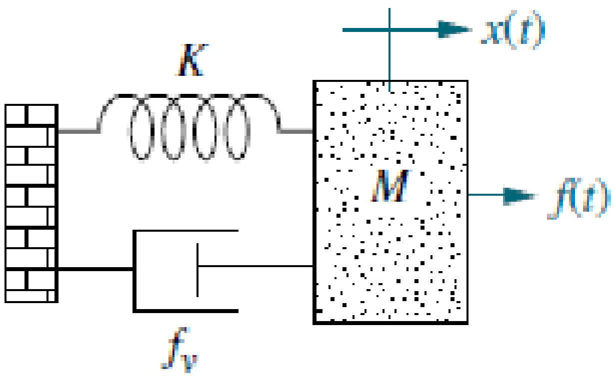
Öteleme Hareketi Yapan Mekanik Sistemlerin TF

Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

	Component	Force-velocity	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
Yay 	Spring 	$f(t) = K \int_0^t v(\tau)d\tau$	$f(t) = Kx(t)$	K
Piston 	Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
	Mass 	$f(t) = M \frac{dv(t)}{dt}$	$f(t) = M \frac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m(newton-seconds/meter), M = kg (kilograms = newton-seconds^2/meter).

Örnek



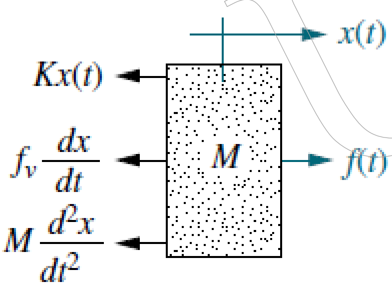
Find the transfer function, $X(s)/F(s)$, for the system of Figure

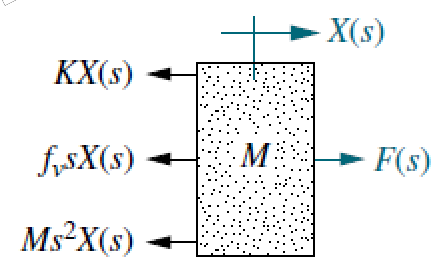
$$M\frac{d^2x(t)}{dt^2} + f_v\frac{dx(t)}{dt} + Kx(t) = f(t)$$

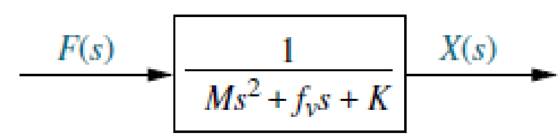
$$Ms^2X(s) + f_v sX(s) + KX(s) = F(s)$$

$$(Ms^2 + f_v s + K)X(s) = F(s)$$

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K}$$

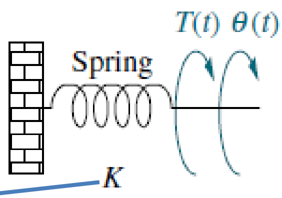
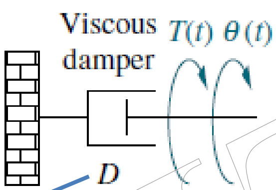
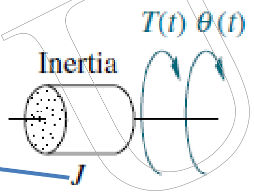






Dönme Hareketi Yapan Mekanik Sistemlerin TF

Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

Component	Torque-angular velocity	Torque-angular displacement	Impedence $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\dfrac{d\theta(t)}{dt}$	Ds
	$T(t) = J\dfrac{d\omega(t)}{dt}$	$T(t) = J\dfrac{d^2\theta(t)}{dt^2}$	Js^2

Yay sabiti

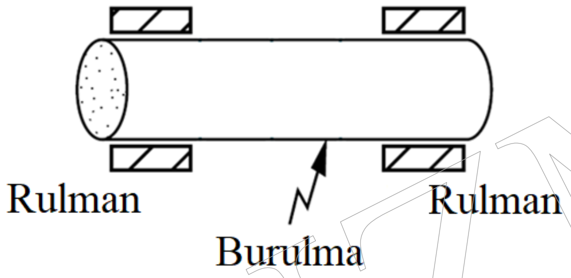
Sönümleme sabiti

Atalet momenti

Note: The following set of symbols and units is used throughout this book: $T(t)$ – N-m (newton-meters), $\theta(t)$ – rad(radians), $\omega(t)$ – rad/s(radians/second), K – N-m/rad(newton- meters/radian), D – N-m-s/rad (newton- meters-seconds/radian). J – kg-m^2(kilograms-meters^2 – newton-meters-seconds^2/radian).

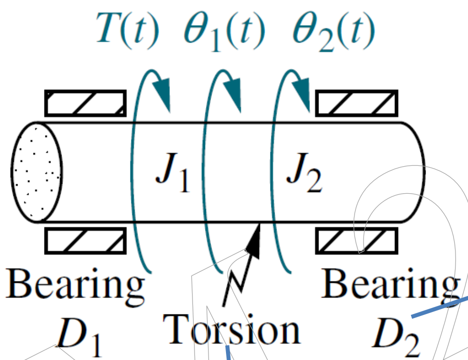
Örnek

- İki ucundan rulmana takılmış ve burulmaya maruz kalan bir şaft aşağıdaki şekilde verilmiştir.
- Şaftın sol tarafına bir tork (dönme momenti) uygulanarak sağ taraftaki dönme ölçülmektedir.
- Dönme hareketi yapan bu mekanik sistem için transfer fonksiyonunu bulunuz.



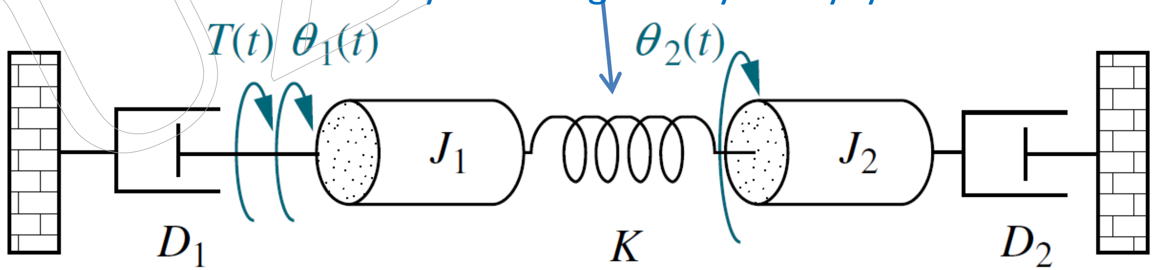
Çözüm

- Sistemin modeli aşağıdaki şekildeki gibi oluşturulursa transfer fonksiyonu:
  $G(s) = \theta_2(s)/T(s)$

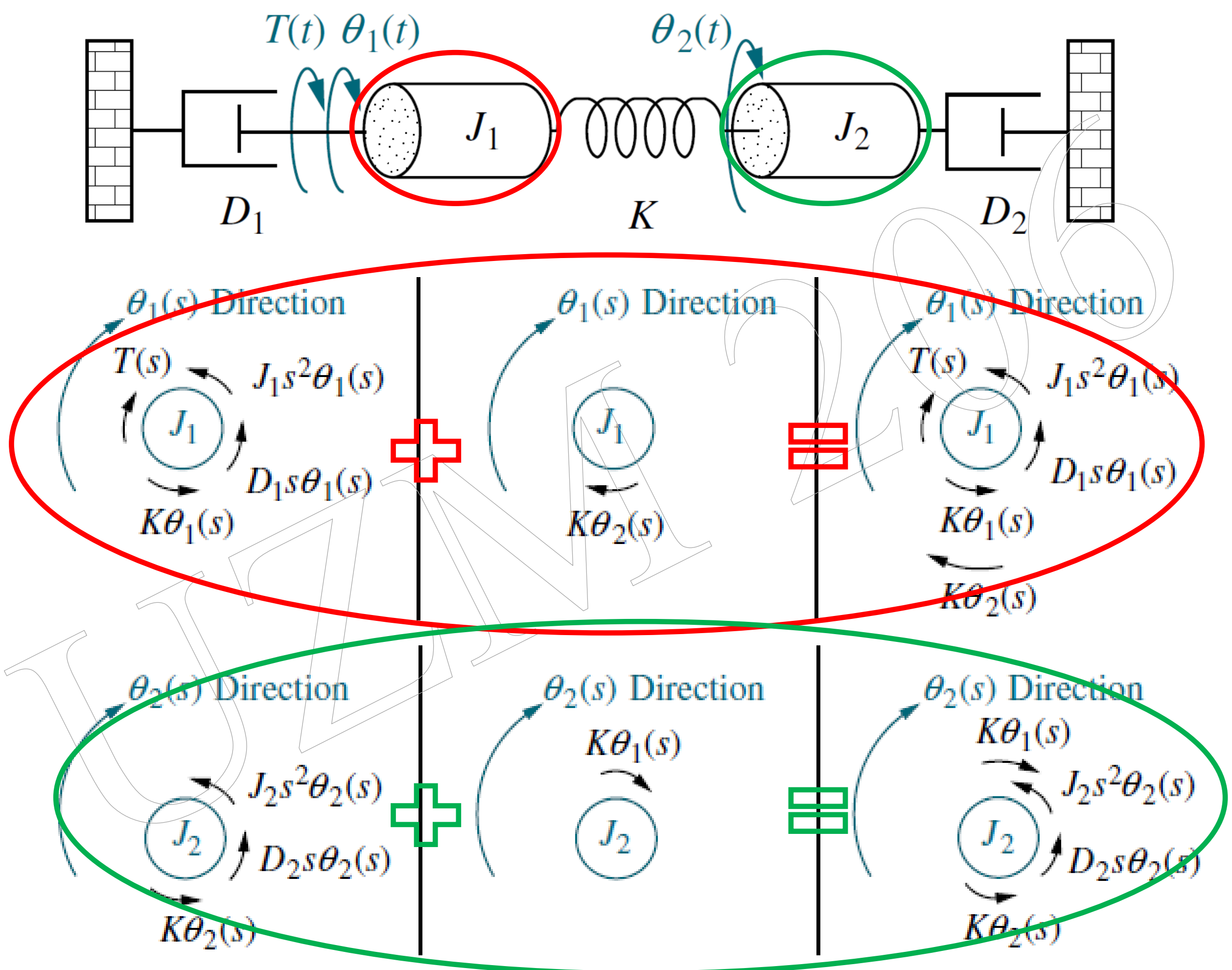


Rulmanla tutturulmuş şaftın dönüşünde meydana gelen sönümleme

Dağınık olan (her yerde bulunan burulmanın) merkezde olduğu ve iki tarafın dönme farkından kaynaklandığı varsayılırsa yay ile modellenebilir.



Çözüm devamı…



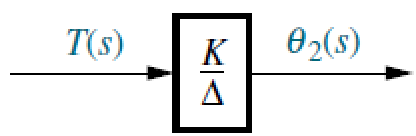
Çözüm devamı…

$$(J_1s^2 + D_1s + K)\underline{\theta_1(s)} \qquad\qquad - K\underline{\theta_2(s)} = T(s)$$

$$-K\underline{\theta_1(s)} + (J_2s^2 + D_2s + K)\underline{\theta_2(s)} = 0$$

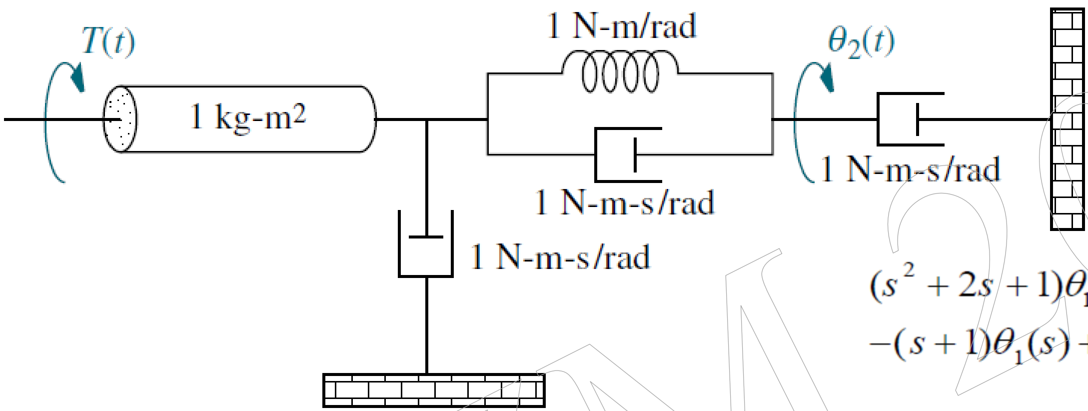
$$\frac{\theta_2(s)}{T(s)} = \frac{K}{\Delta}$$

$$\Delta = \begin{vmatrix} (J_1s^2 + D_1s + K) & -K \\ -K & (J_2s^2 + D_2s + K) \end{vmatrix}$$



Örnek

- Dönme hareketi yapan aşağıdaki sistemin $G(s) = \theta_2(s)/T(s)$ transfer fonksiyonunu bulunuz.



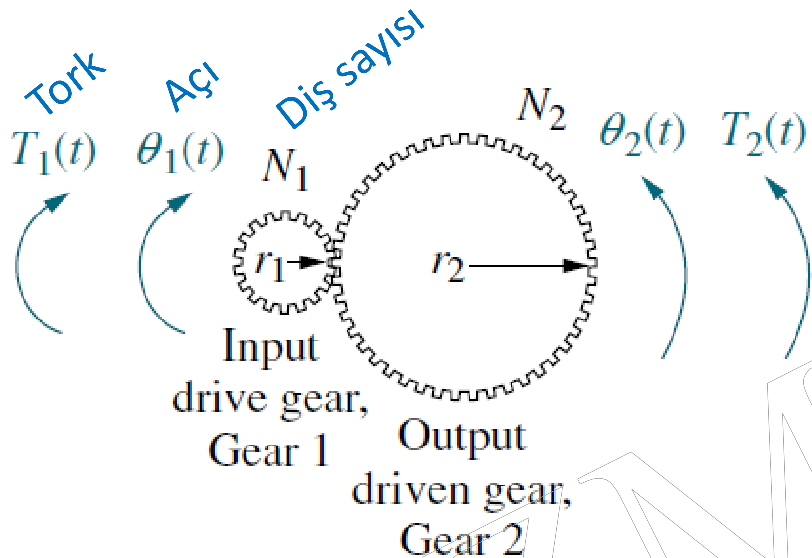
$$(s^2 + 2s + 1)\theta_1(s) - (s+1)\theta_2(s) = T(s)$$
$$-(s+1)\theta_1(s) + (2s+1)\theta_2(s) = 0$$

$$\theta_2(s) = \frac{\begin{vmatrix} (s^2+2s+1) & T(s) \\ -(s+1) & 0 \end{vmatrix}}{\begin{vmatrix} (s^2+2s+1) & -(s+1) \\ -(s+1) & (2s+1) \end{vmatrix}} = \frac{T(s)}{2s(s+1)}$$

Ödev: Bu sistem 1 Nm ile sabit bir şekilde döndürülmeye başlanırsa θ_2 açısı zamanla nasıl değişir? Grafiğini çiziniz.

$$\frac{\theta_2(s)}{T(s)} = \frac{1}{2s(s+1)}$$

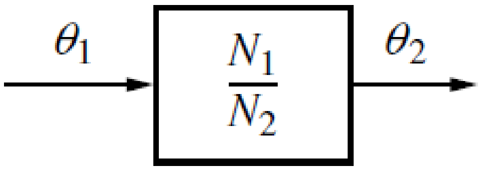
Dişli İçeren Sistemlerin TF

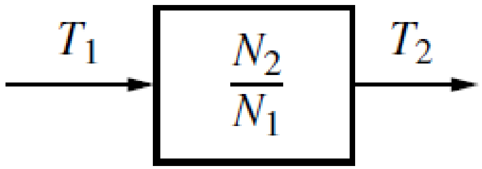


$$r_1\theta_1 = r_2\theta_2$$

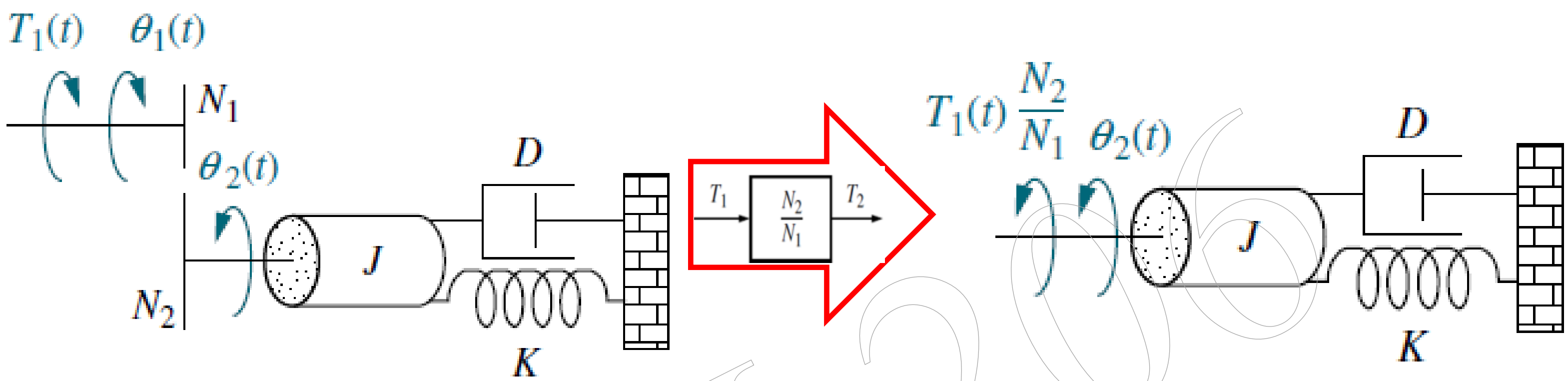
$$\frac{r_1}{r_2} = \frac{\theta_2}{\theta_1} = \frac{N_1}{N_2}$$

$$T_1\theta_1 = T_2\theta_2$$





Dişli İçeren Sistemlerin TF



$$(Js^2 + Ds + K)\theta_2(s) = T_1(s)\frac{N_2}{N_1}$$

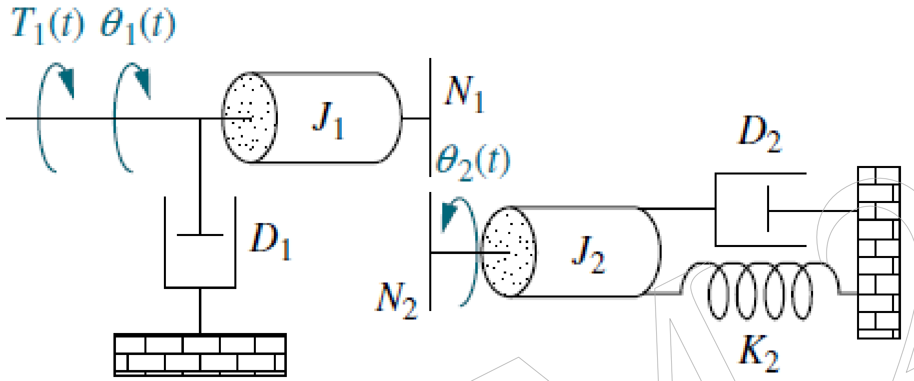
$$(Js^2 + Ds + K)\frac{N_1}{N_2}\theta_1(s) = T_1(s)\frac{N_2}{N_1}$$

$$\left(\frac{N_2}{N_1}\right)^2$$

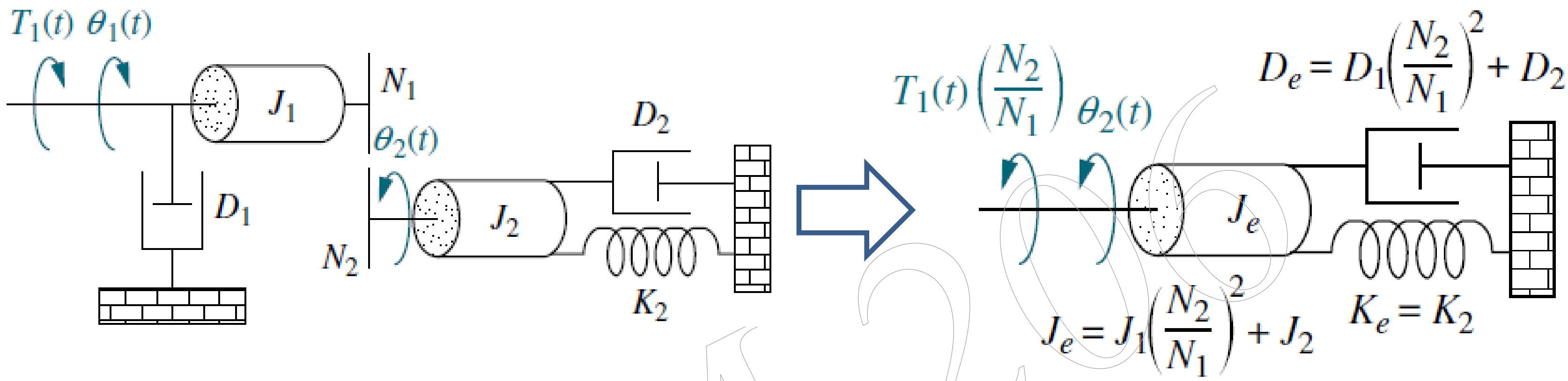
$$\left[J\left(\frac{N_1}{N_2}\right)^2 s^2 + D\left(\frac{N_1}{N_2}\right)^2 s + K\left(\frac{N_1}{N_2}\right)^2\right]\theta_1(s) = T_1(s)$$

Örnek

- Aşağıdaki şekilde verilen sistem için $\theta_2(s)/T_1(s)$ transfer fonksiyonunu bulunuz.



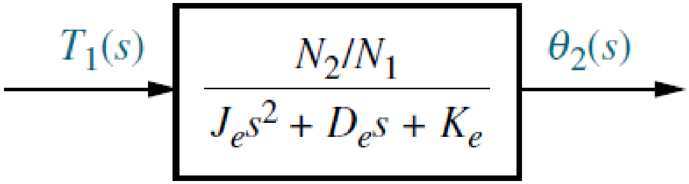
Çözüm



$$(J_e s^2 + D_e s + K_e)\theta_2(s) = T_1(s)\frac{N_2}{N_1}$$

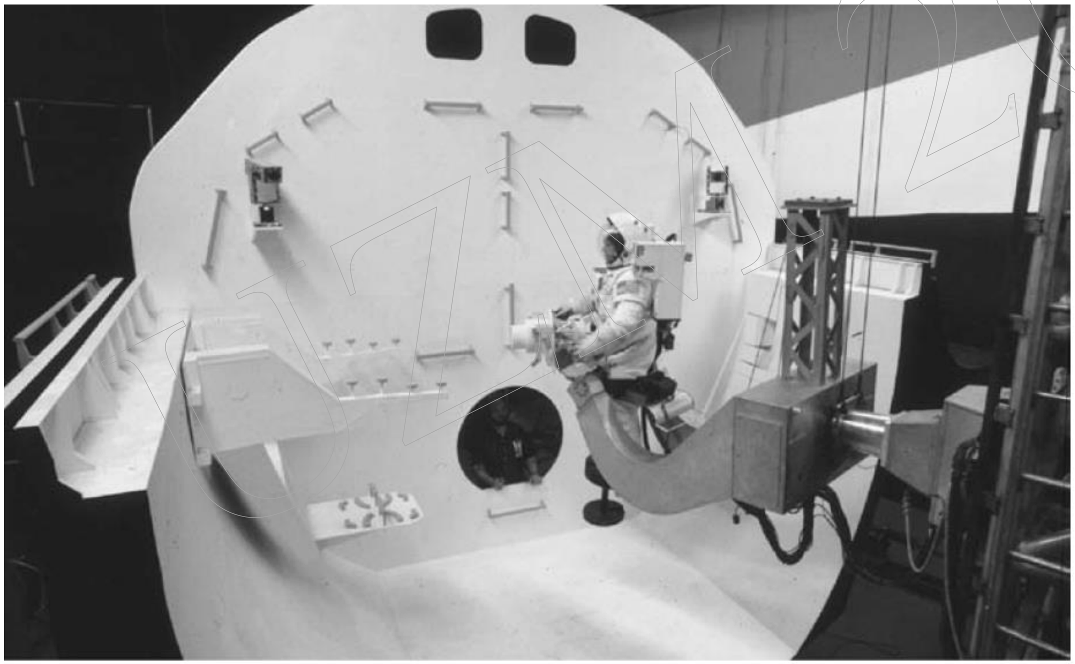
$$J_e = J_1\left(\frac{N_2}{N_1}\right)^2 + J_2; \quad D_e = D_1\left(\frac{N_2}{N_1}\right)^2 + D_2; \quad K_e = K_2$$

$$G(s) = \frac{\theta_2(s)}{T_1(s)} = \frac{N_2/N_1}{J_e s^2 + D_e s + K_e}$$



Elektromekanik Sistemlerin TF

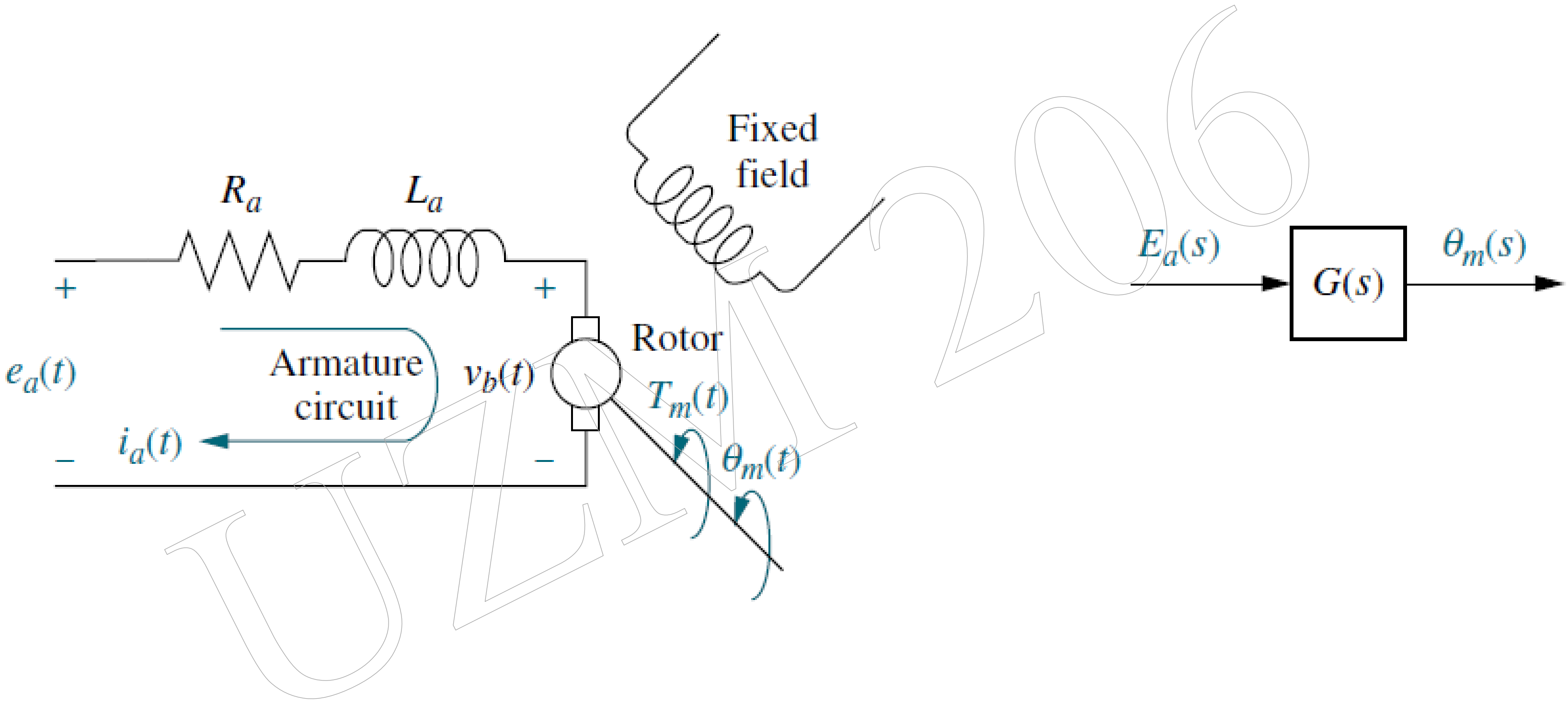
- Elektromekanik sistemler hem elektriksel hem de mekanik bileşenler içerir.
- Elektromekanik sistemlerin bazı uygulamaları: robotik sistemler, güneş ve yıldız takipçileri, DVD sürücü…
- Sistemlerde önemli elektromekanik bileşenlerden biri **motor**dur. Girişine uygulanan gerilime göre döner. Yani elektrik girişiyle mekanik çıkış üretir.



NASA flight simulator robot arm with electromechanical control system components.

Elektromekanik Sistemlerin TF

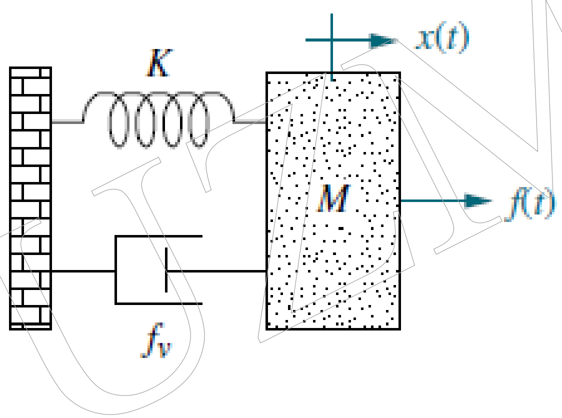
- Elektromekanik sistemin özel bir türü, armatür kontrollü DC **servo motor**dur.

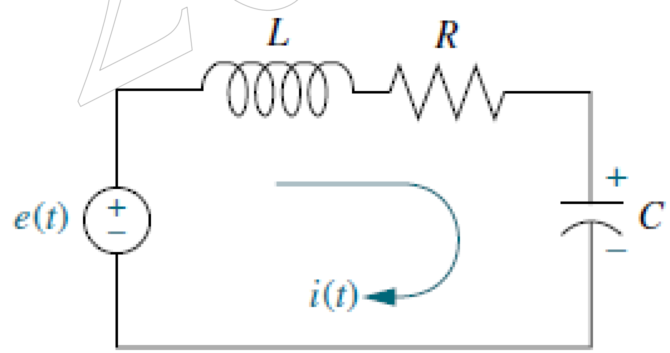


Benzerlik

- Mekanik sistemin hareket denklemleri ile elektrik sistemlerinin ki karşılaştırıldıklarında arada önemli benzerlikler olduğu görülür. Mekanik sistem ile elektrik devrelerindeki **göz/çevrim** denklemleri arasındaki benzerlik **seri**, Mekanik sistem ile elektrik devrelerindeki **düğüm** denklemleri arasındaki benzerlik **paralel** olarak isimlendirilir.

Seri Benzerlik:





$$(Ms^2 + f_v s + K)X(s) = F(s)$$

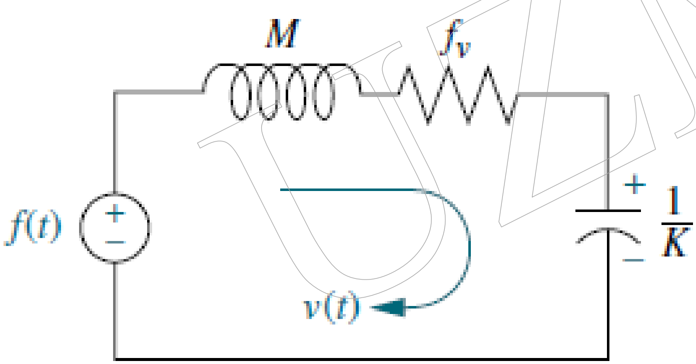
$$\left(Ls + R + \frac{1}{Cs}\right)I(s) = E(s)$$

Seri Benzerlik

- Mekanik sistem denkleminde **yer değiştirme** (X) yerine **hız** (V) alındığında benzerlik görülür. Bunun için eşitliğin sol tarafı s ile çarpılıp bölünürse:

$$\frac{Ms^2 + f_v s + K}{s}\boxed{sX(s)} = \left(Ms + f_v + \frac{K}{s}\right)\boxed{V(s)} = F(s)$$

$$\left(Ls + R + \frac{1}{Cs}\right)I(s) = E(s)$$



mass = M ⟶ inductor = M henries

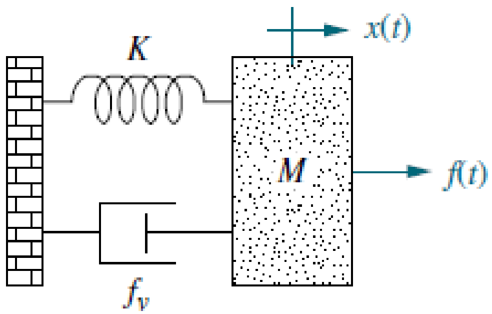
viscous damper = f_v ⟶ resistor = f_v ohms

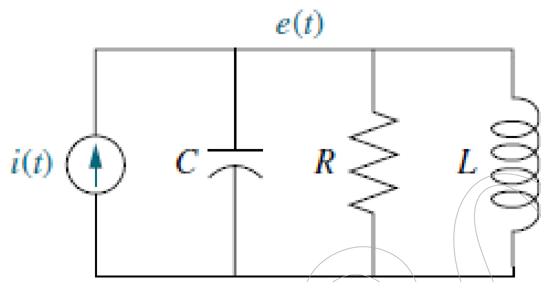
spring = K ⟶ capacitor = $\frac{1}{K}$ farads

applied force = $f(t)$ ⟶ voltage source = $f(t)$

velocity = $v(t)$ ⟶ mesh current = $v(t)$

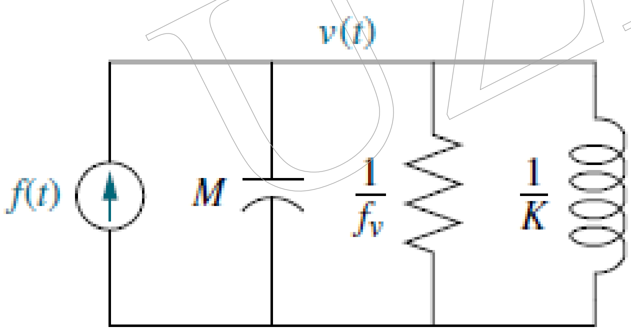
Paralel Benzerlik





$$\left(Cs + \frac{1}{R} + \frac{1}{Ls}\right)E(s) = I(s)$$

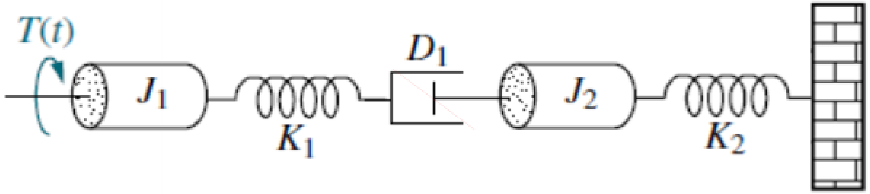
$$\frac{Ms^2 + f_v s + K}{s} sX(s) = \left(Ms + f_v + \frac{K}{s}\right)V(s) = F(s)$$



mass = M	$\longrightarrow$	capacitor	= M farads
viscous damper = f_v	$\longrightarrow$	resistor	= $\frac{1}{f_v}$ ohms
spring = K	$\longrightarrow$	inductor	= $\frac{1}{K}$ henries
applied force = $f(t)$	$\longrightarrow$	current source	= $f(t)$
velocity = $v(t)$	$\longrightarrow$	node voltage	= $v(t)$

Örnek

- Şekildeki dönme hareketli sistemin seri benzerini bulunuz.



Çözüm:

$$(J_1 s + D_1 + \frac{K_1}{s})\Omega_1(s) - (D_1 + \frac{K_1}{s})\Omega_2(s) = T(s)$$

$$-(D_1 + \frac{K_1}{s})\Omega_1(s) + (J_2 s + D_1 + \frac{(K_1 + K_2)}{s})\Omega_2(s) = 0$$