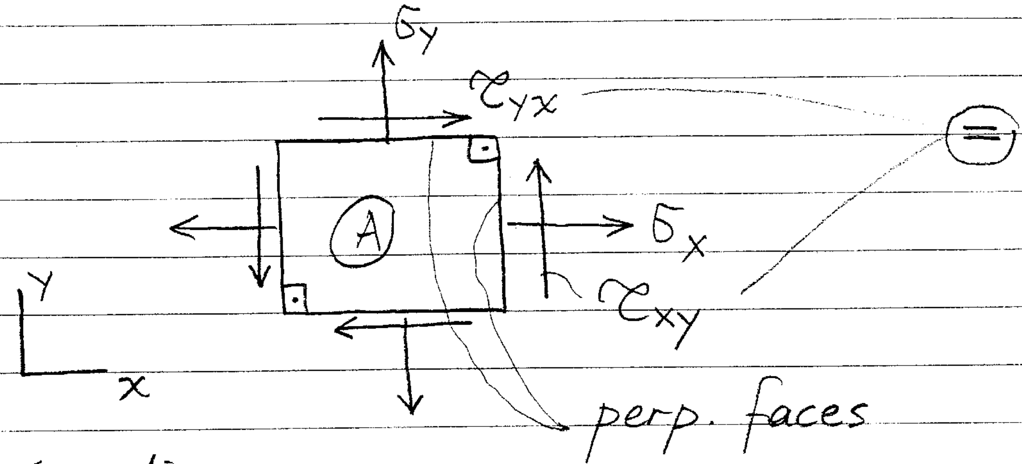
<u>FBD for (A):</u>



in x-dir.   in y-dir.

σ_x, σ_y $\rightarrow$ normal stresses (NS)

$\oplus$ if directed outwards (T)
$\ominus$ " " inwards (C)

τ_{xy} $\rightarrow$ shear stress (SS) acting on x-f in y-dir.

face   dir.

τ_{yx} $\rightarrow$ SS acting on y-f in x-dir.

τ_{xy} is $\oplus$ if it acts in $\oplus$-y dir.
τ_{yx} " " " " " " " x "

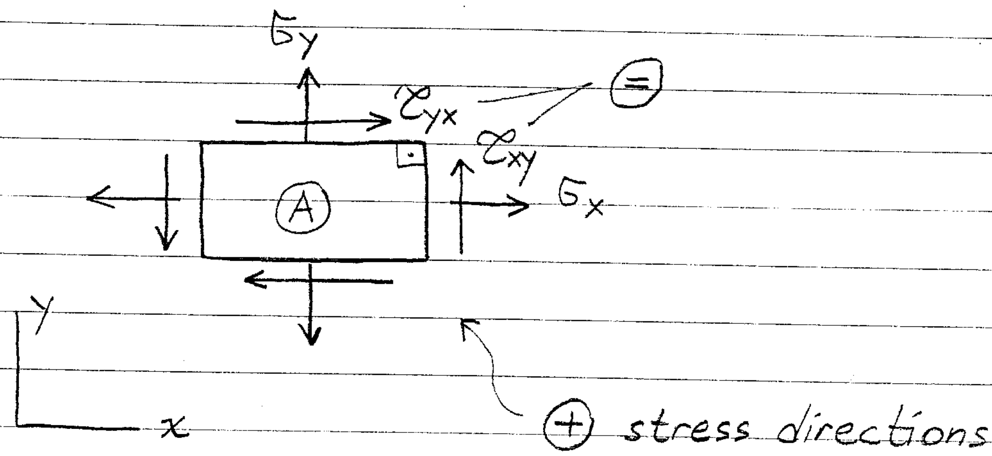
This sign convention holds for $\oplus$-f's.

$\Rightarrow$ The opposite sign convention holds for $\ominus$-f's.

$$\boxed{\tau_{xy} = \tau_{yx}}$$

Stress components:

$\left.\begin{matrix}\sigma_x \\ \sigma_y\end{matrix}\right\}$ NS $\qquad$ $\left.\begin{matrix}\tau_{xy} \\ \tau_{yx}\end{matrix}\right\}$ SS $\qquad \tau_{xy} = \tau_{yx}$

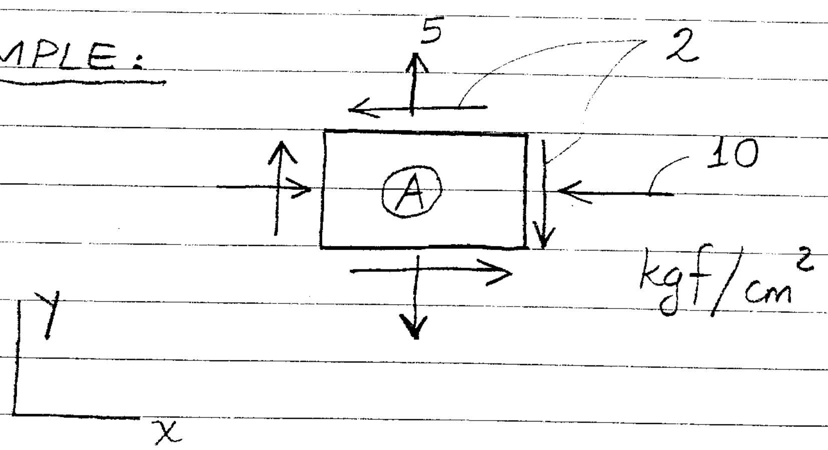


we can express the stress components in matrix form as follows:

$$\underbrace{\underline{\sigma}}_{\text{stress matrix}} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix} \begin{matrix} \text{stress comp.'s of x-f.} \\ \text{stress comp's of y-f.} \end{matrix}$$

(2x2) symmetric matrix

EXAMPLE:

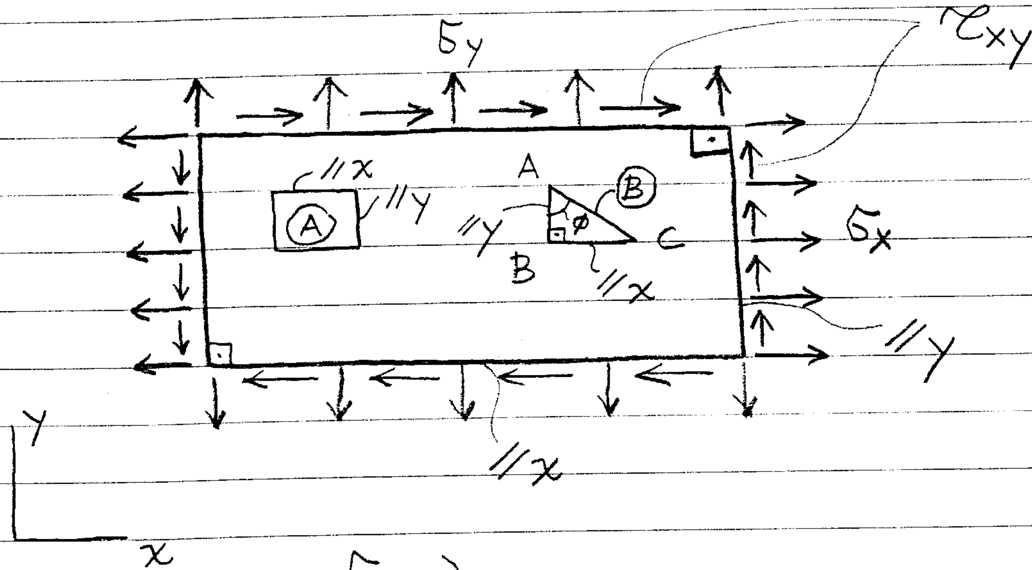


$$\Rightarrow \quad \sigma_x = -10 \quad \text{C}$$
$$\sigma_y = +5 \quad \text{T}$$

$$\tau_{xy} = -2$$

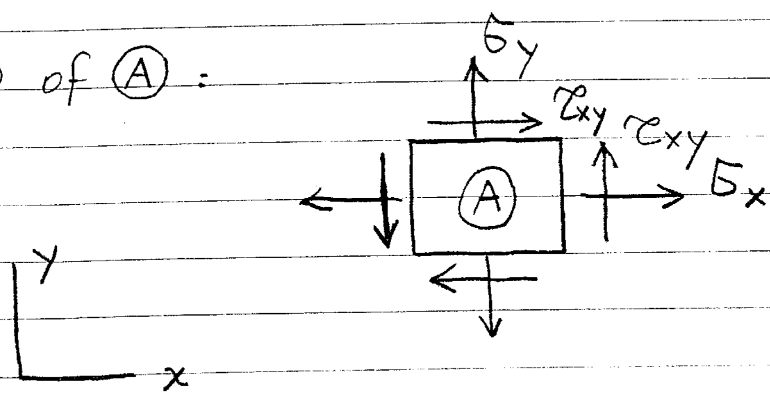
$$\Rightarrow \quad \underline{\sigma} = \begin{bmatrix} -10 & -2 \\ -2 & 5 \end{bmatrix}$$

Consider a rectangular plate whose sides are subjected to uniform stresses:

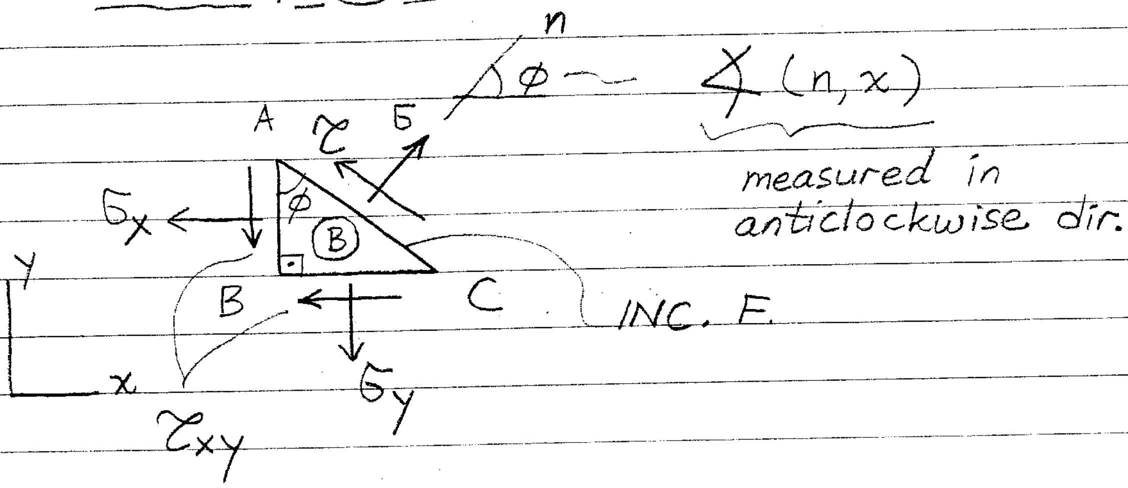


$$\left.\begin{matrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{matrix}\right\} \text{ GIVEN} \Rightarrow \text{KNOWN}$$

FBD of A:



FBD of (B):



wish to find σ, τ in terms of σ_x, σ_y and τ_{xy}. For that use EE:

$t \rightarrow$ thickness of the plate

$\overset{+}{\nearrow} \sum F_\sigma = 0 \leftarrow$ EE in σ-dir

$$\Rightarrow \quad \sigma * \overline{AC} * t - \sigma_y \sin\phi * \overline{BC} * t$$

$$(*) \quad - \sigma_x \cos\phi * \overline{AB} * t - \tau_{xy} \cos\phi * \overline{BC} * t$$

$$- \tau_{xy} \sin\phi * \overline{AB} * t = 0$$

$(*) / \overline{AC}$:

$$\Rightarrow \quad \boxed{\sigma = \sigma_y \sin^2\phi + \sigma_x \cos^2\phi + 2\tau_{xy} \sin\phi \cos\phi} \quad (**)$$

$$\left.\begin{aligned} \sin^2\phi &= \frac{1}{2}(1-\cos 2\phi) \\ \cos^2\phi &= \frac{1}{2}(1+\cos 2\phi) \\ \sin\phi\cos\phi &= \frac{1}{2}\sin 2\phi \end{aligned}\right\} \text{A}$$

A $\rightarrow$ ($**$) :

$$\Rightarrow \boxed{\sigma = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\phi + \tau_{xy}\sin 2\phi} \quad (1)$$

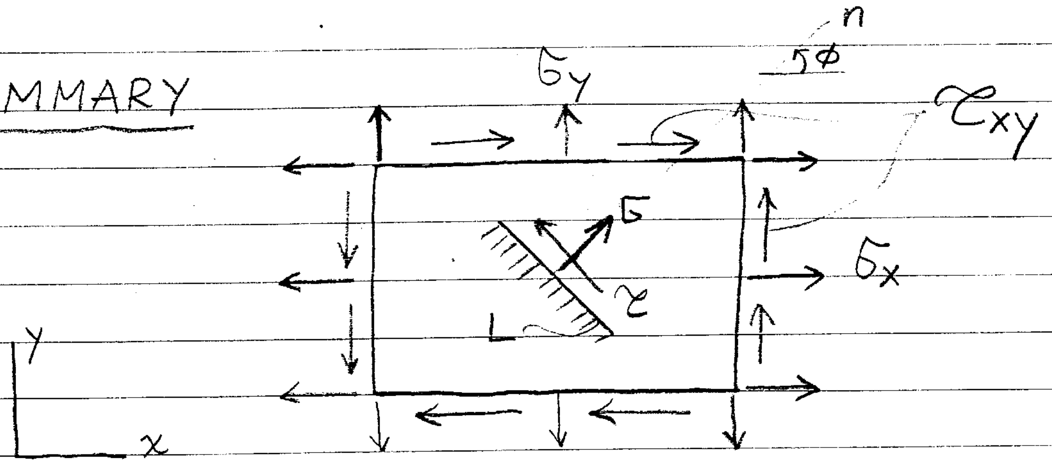
Now write:

$$\oplus\nwarrow \sum F_\tau = 0 \leftarrow \text{EE in } \tau\text{-dir.}$$

$$\Rightarrow \boxed{\tau = -\frac{\sigma_x-\sigma_y}{2}\sin 2\phi + \tau_{xy}\cos 2\phi} \quad (2)$$

(1) & (2) determine σ & τ in terms of σ_x, σ_y, τ_{xy}

SUMMARY



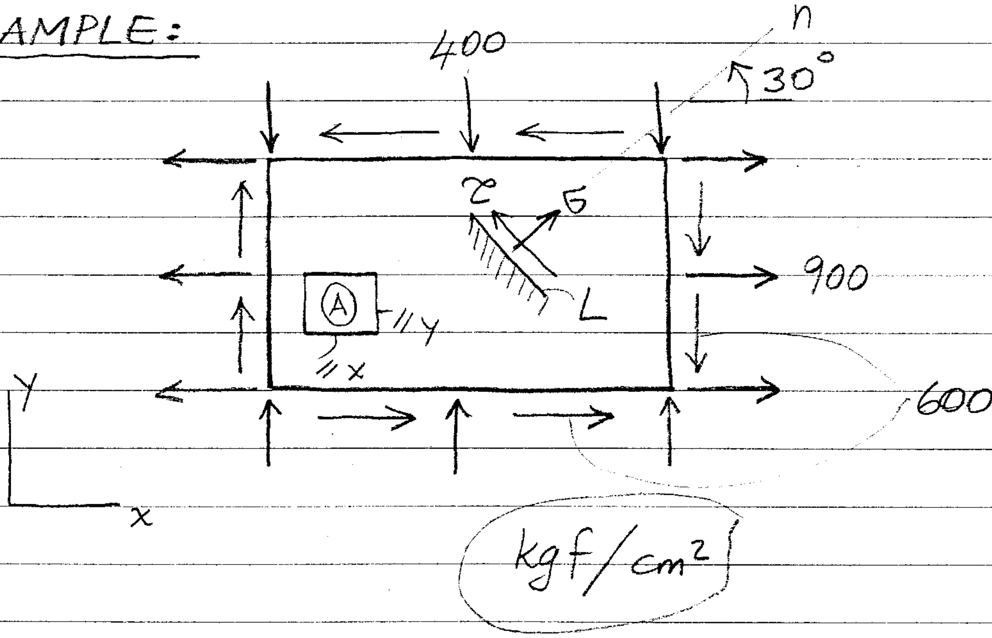
$\sigma_x, \sigma_y, \tau_{xy} \rightarrow$ GIVEN

σ, τ acting on L:

$$\sigma = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\phi + \tau_{xy} \sin 2\phi \quad (1)$$

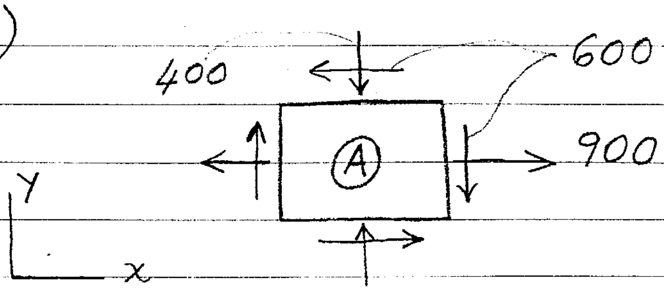
$$\tau = -\frac{\sigma_x - \sigma_y}{2} \sin 2\phi + \tau_{xy} \cos 2\phi \quad (2)$$

EXAMPLE:



we wish to determine the stresses acting
a) on (A)
b) on L

a)



b) $\left.\begin{aligned}\sigma_x &= 900\\ \sigma_y &= -400\\ \tau_{xy} &= -600\end{aligned}\right\}$ use ① and ②

$$\phi = 30^\circ \Rightarrow 2\phi = 60^\circ$$

$$\Rightarrow \cos 2\phi = \frac{1}{2}, \quad \sin 2\phi = \frac{\sqrt{3}}{2}$$

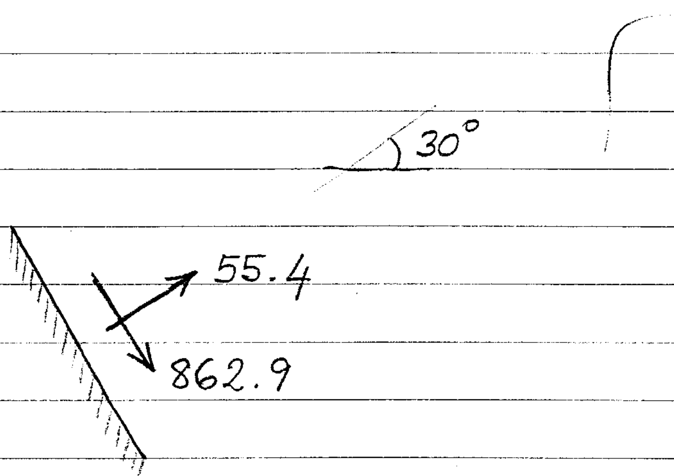
$$\Rightarrow \frac{\sigma_x + \sigma_y}{2} = 250, \quad \frac{\sigma_x - \sigma_y}{2} = 650$$

$$① \Rightarrow \sigma = 250 + (650) * \frac{1}{2} + (-600) * \frac{\sqrt{3}}{2}$$

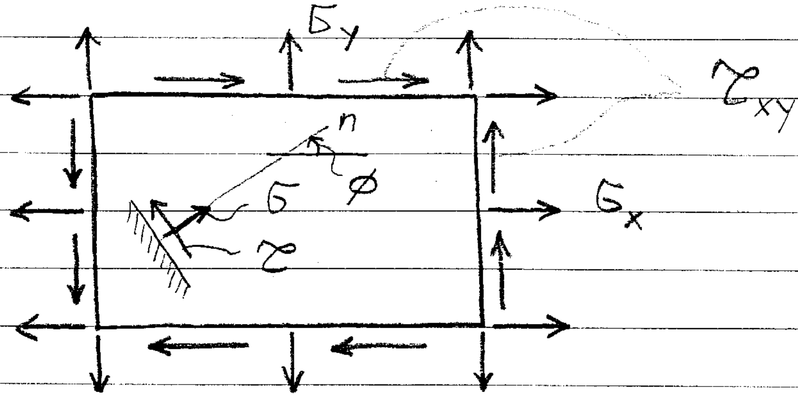
$$\Rightarrow \sigma \cong 55.4 \ \text{kgf/cm}^2 \ ⓣ$$

$$② \Rightarrow \tau = -(650) * \frac{\sqrt{3}}{2} + (-600) * \frac{1}{2}$$

$$\Rightarrow \tau = -862.9 \ \text{kgf/cm}^2$$



MC FOR PLANE STATE OF STRESS

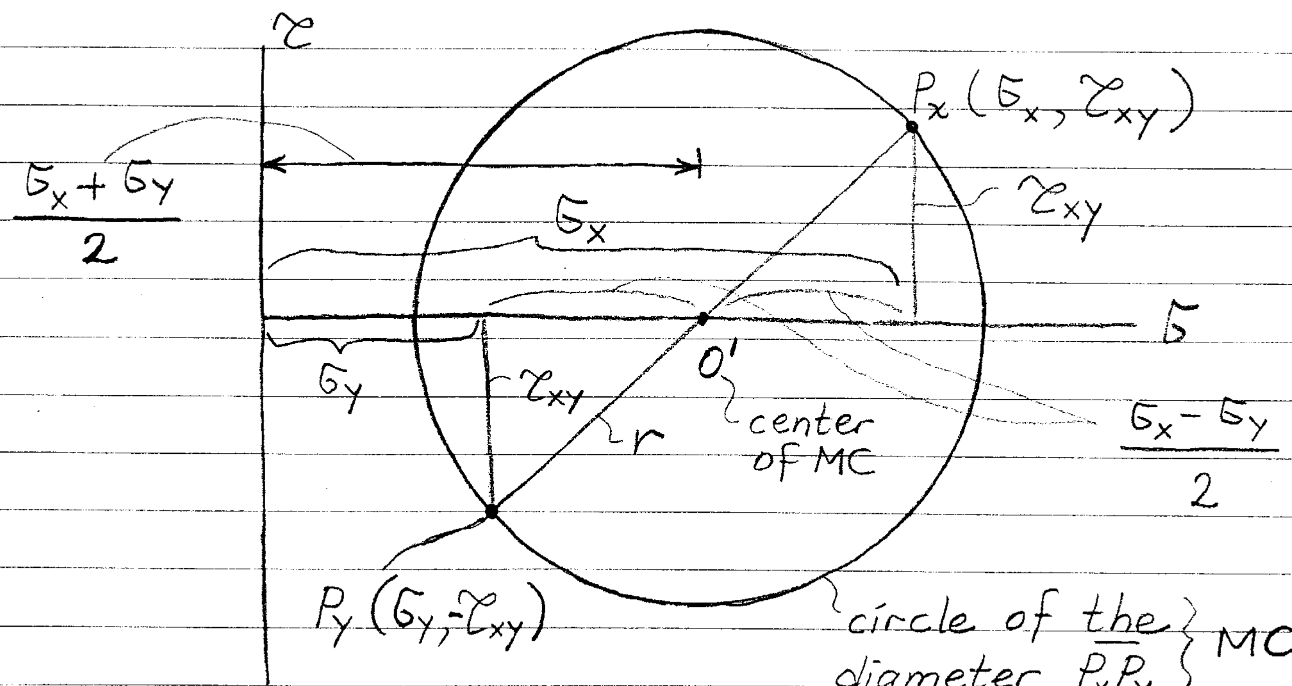


$$\left.\begin{matrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{matrix}\right\}\text{ GIVEN}$$

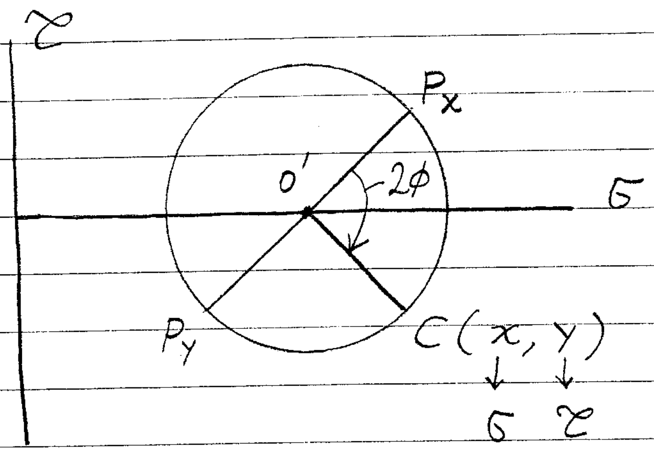
wish to find σ and τ by MC.

For that use the following steps.

1) Construct MC:



2) Take the angle 2ϕ from $\overline{O'P_x}$ in clockwise dir.



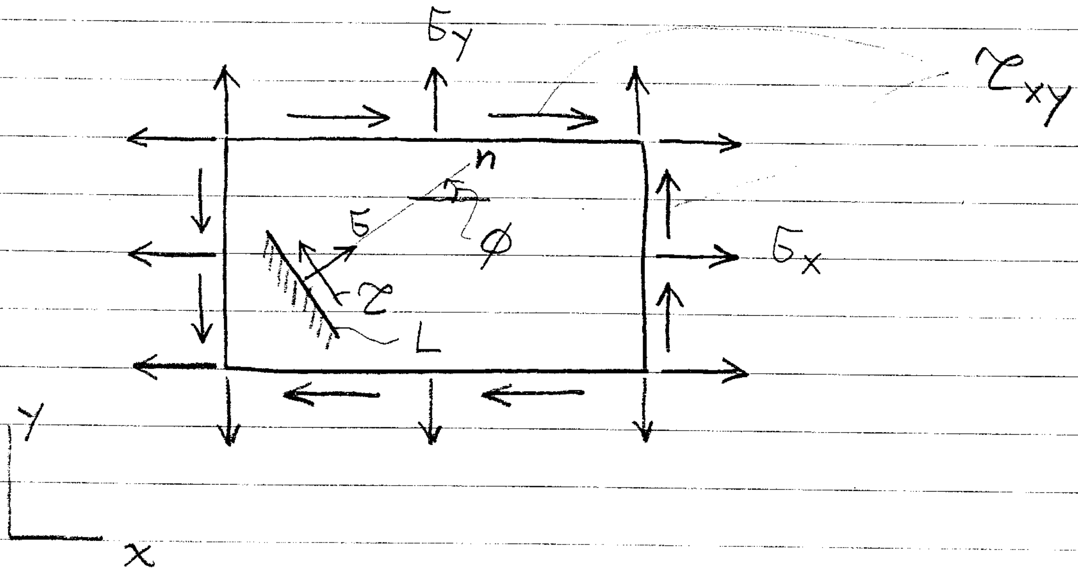
This determines a pt "C" on MC

3) by computing the coordinates of C find σ and τ acting on L

$$C \rightarrow (x, y)$$
$$\downarrow \quad \downarrow$$
$$\sigma \quad \tau$$

PRINCIPAL STRESSES

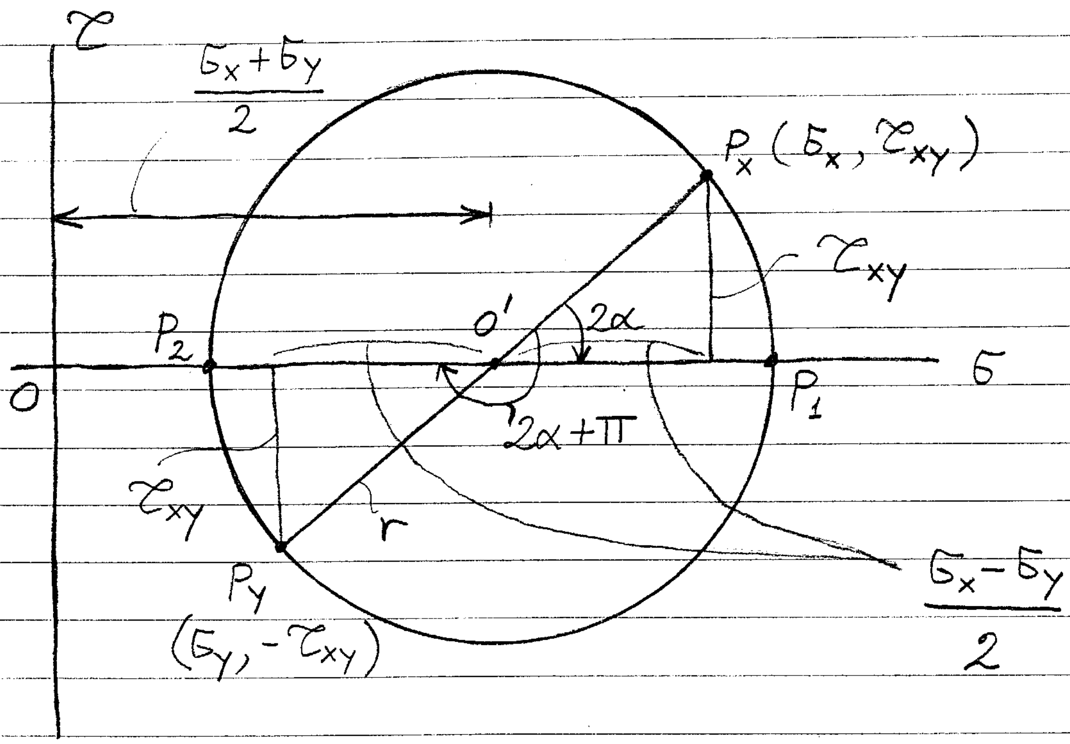
- The plane on which $\tau = 0$ is called principal plane – PP
- The normal stresses acting on PP are called principal stresses – PS
- The element formed by PP's is called principal element – PE



$$\left.\begin{array}{l}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{array}\right\}\text{KNOWN}$$

wish to find PS's.

Use MC :



$$r = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\operatorname{Tan} 2\alpha = \frac{\tau_{xy}}{\frac{\sigma_x - \sigma_y}{2}} = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\operatorname{Sin} 2\alpha = \frac{\tau_{xy}}{r}$$

these two eqs. determine "α" uniquely

At P_1 and P_2 $\quad \tau = 0$

$\Rightarrow P_1, P_2$ will determine PP's and PS's

$\underline{P_1:}$

$$2\phi = 2\alpha \Rightarrow \phi = \alpha$$

$$\Rightarrow \tau = 0$$

$$\sigma = \overline{OO'} + r \qquad \frac{\sigma_x + \sigma_y}{2}$$

$$\Rightarrow \boxed{\sigma = \frac{\sigma_x + \sigma_y}{2} + r} \quad \sigma_1$$

$\underline{P_2:}$

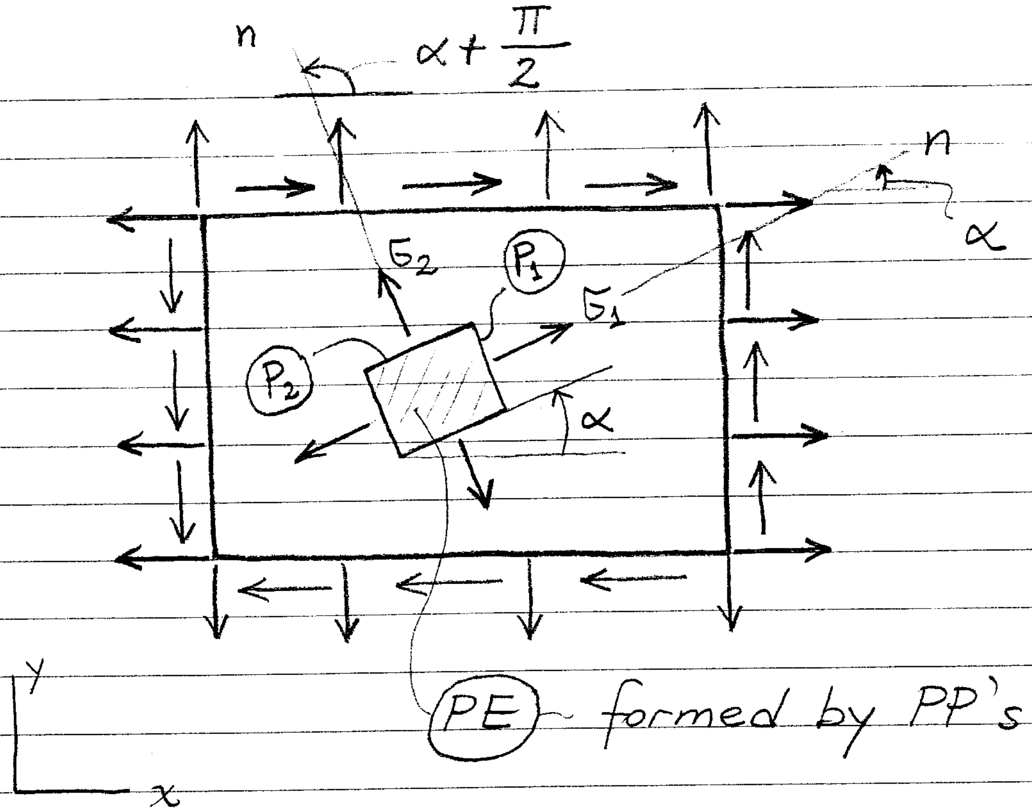
$$2\phi = 2\alpha + \pi \Rightarrow \phi = \alpha + \frac{\pi}{2}$$

$$\Rightarrow \tau = 0$$

$$\sigma = \overline{OO'} - r \qquad \frac{\sigma_x + \sigma_y}{2}$$

$$\Rightarrow \boxed{\sigma = \frac{\sigma_x + \sigma_y}{2} - r}$$

$$\sigma_2$$



$$\sigma_1, \sigma_2 \rightarrow PS's$$

SUMMARY

To compute PE and PS's use the following steps:

1) Compute $r = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

2) Compute the angle α from:

$$\tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}, \quad \sin 2\alpha = \frac{\tau_{xy}}{r}$$

$$\Rightarrow \alpha$$

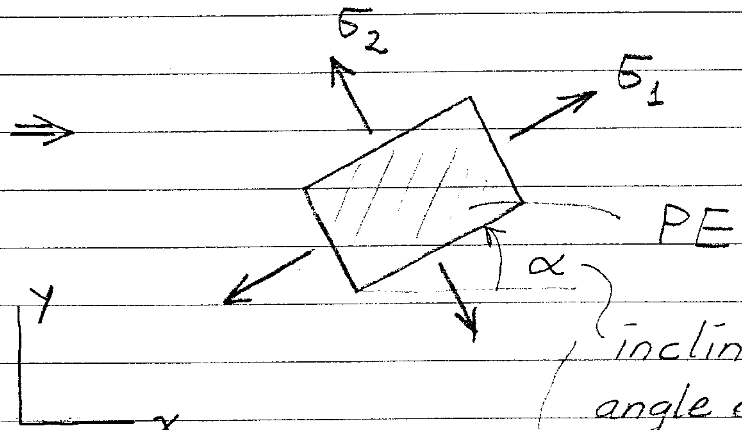
$\rightarrow$

3) Compute PS's by:

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + r$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - r$$

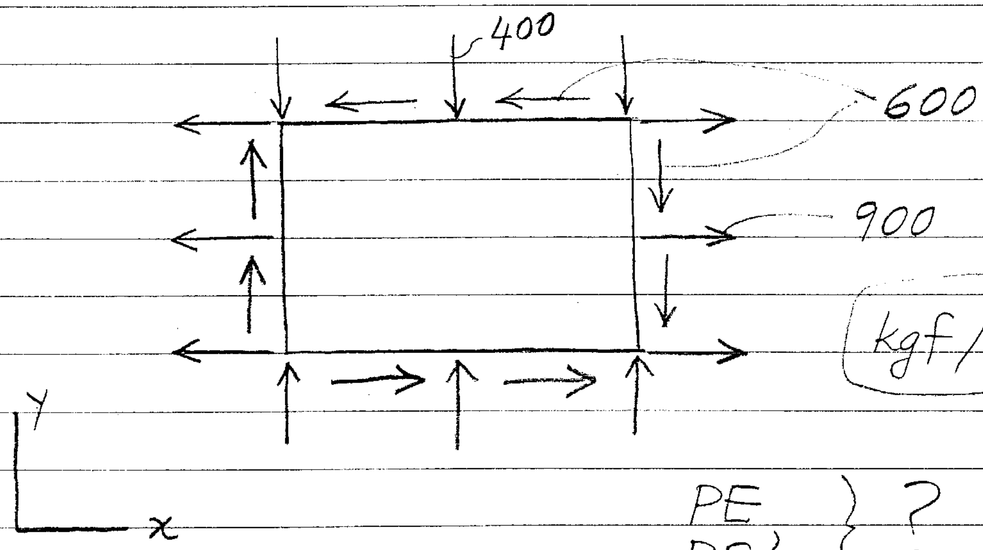
$\Rightarrow$



inclination angle of PE from x-axis

measured in anticlockwise dir.

EXAMPLE: (Previous example)



kgf/cm^2

PE
PS's } ?

$$\sigma_x = 900 \quad \text{T}$$
$$\sigma_y = -400 \quad \text{C}$$
$$\tau_{xy} = -600$$

1) $$r = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\Rightarrow \boxed{r \cong 884.6 \ \mathrm{kgf/cm^2}}$$

2) $$\tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2*(-600)}{1300}$$

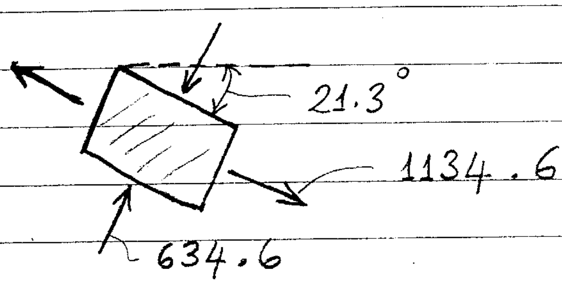
$$\Rightarrow \tan 2\alpha \cong -0.92 \Rightarrow 2\alpha = -42.6^\circ \overset{?}{\cancel{+\pi}}$$

$$\sin 2\alpha = \frac{\overset{-}{\tau_{xy}}}{\underset{+}{r}} < 0$$

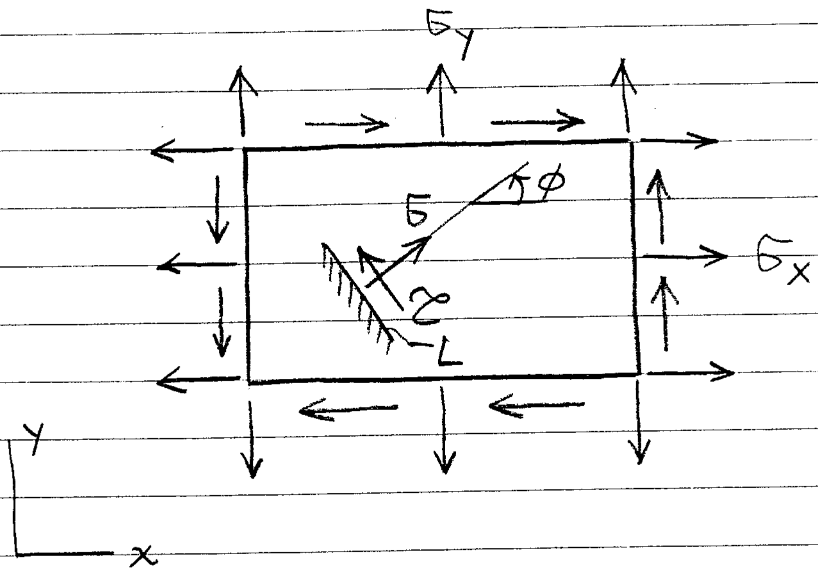
$$\Rightarrow \boxed{\alpha = -21.3^\circ}$$

3) $$\sigma_1 = \overbrace{\frac{\sigma_x + \sigma_y}{2}}^{250} + \underbrace{r}_{884.6} = 1134.6 \quad \text{T}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - r = -634.6 \quad \text{C}$$

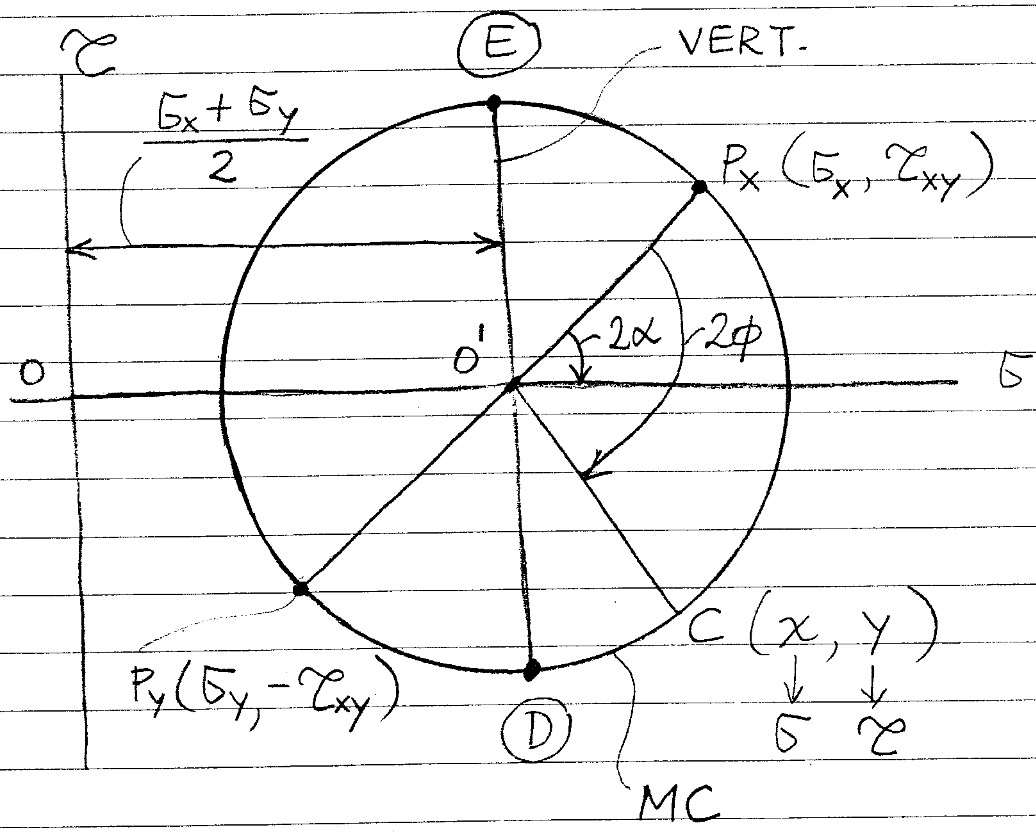


τ_{max} FOR PLANE STATE OF STRESS



$\sigma_x, \sigma_y, \tau_{xy} \rightarrow$ GIVEN

Construct MC:



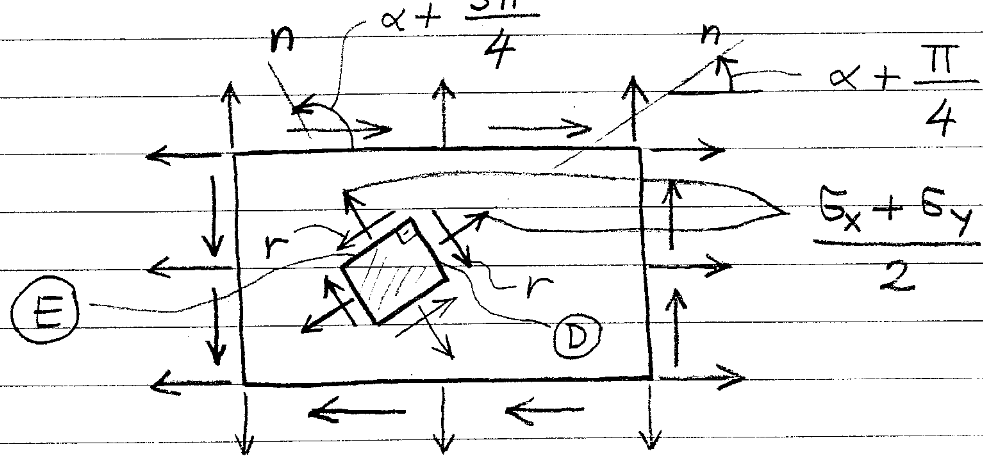
$\Rightarrow \tau$ will be max at D & E

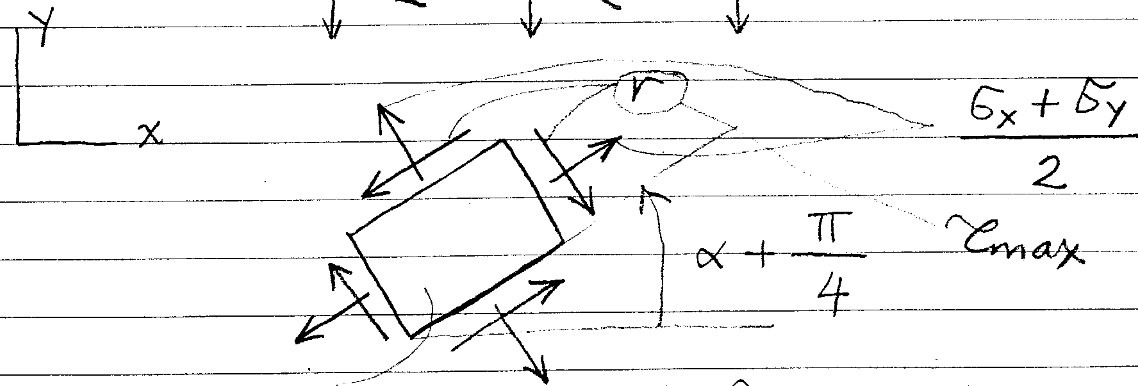
$\Rightarrow \tau_{max}$ will occur at D & E

$$\tau_{max} = \max |\tau|$$

$$\Rightarrow \boxed{\tau_{max} = r} \quad \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

D	E
$2\phi = 2\alpha + \frac{\pi}{2}$	$2\phi = 2\alpha + \frac{3\pi}{2}$
$\Rightarrow \phi = \alpha + \frac{\pi}{4}$	$\Rightarrow \phi = \alpha + \frac{3\pi}{4}$
$\sigma = \frac{\sigma_x + \sigma_y}{2}$	$\sigma = \frac{\sigma_x + \sigma_y}{2}$
$\tau = -r$	$\tau = +r$





The element on which τ_{max} acts.

EXAMPLE: (Previous example)

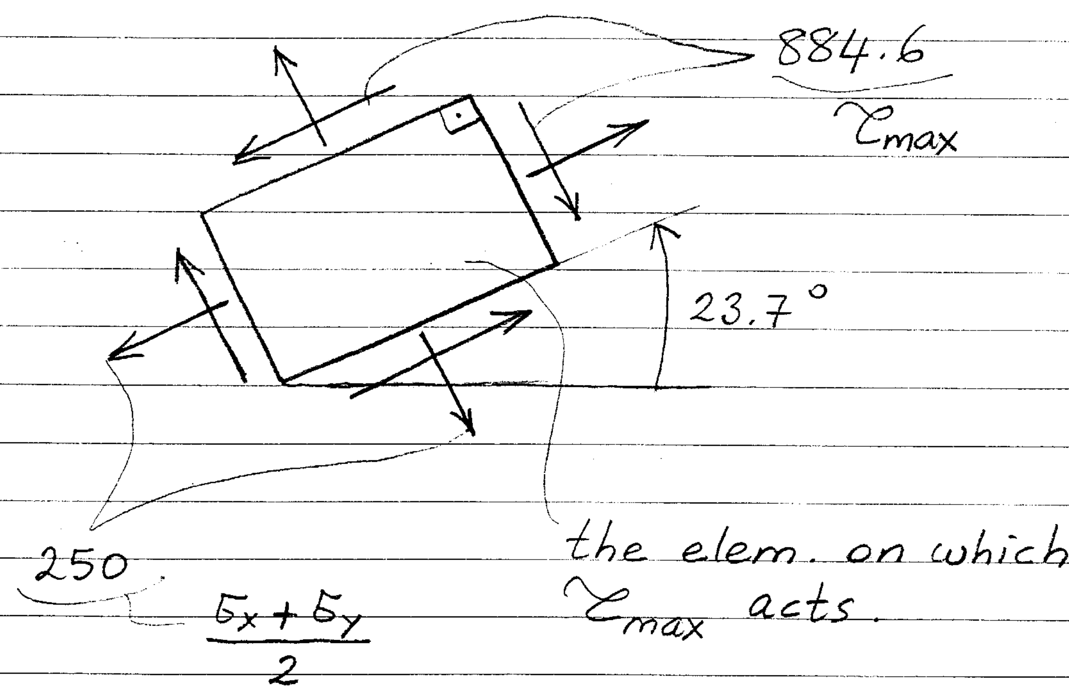
$$\sigma_x = 900$$
$$\sigma_y = -400 \qquad \boxed{kgf/cm^2}$$
$$\tau_{xy} = -600$$

τ_{max} and the elem. on which it acts?

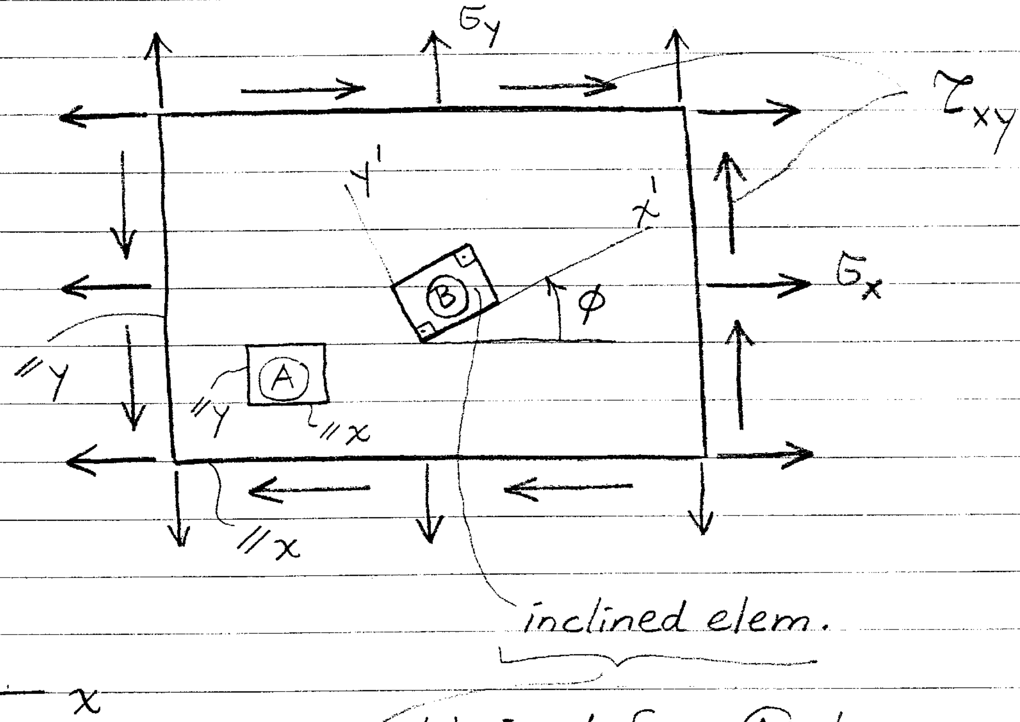
$$\left.\begin{array}{l} r = 884.6 \ \ kgf/cm^2 \\ \alpha = -21.3^\circ \end{array}\right\} \text{computed previously}$$

$$\Rightarrow \quad \boxed{\tau_{max} = r = 884.6 \ \ kgf/cm^2}$$

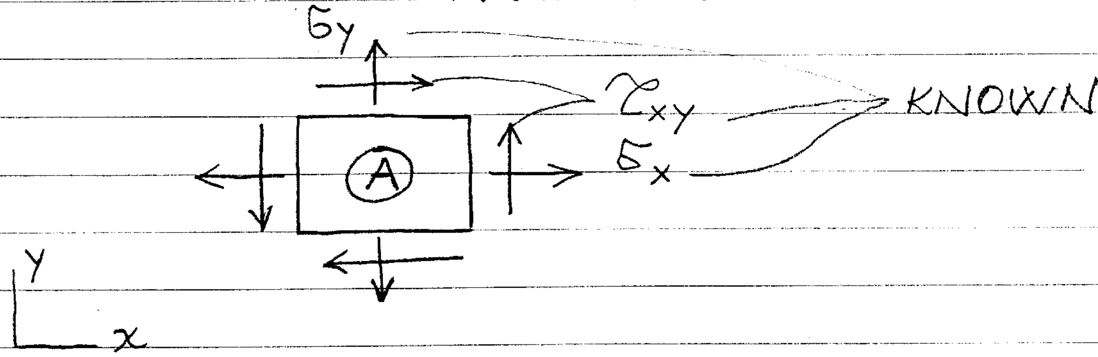
$$\underbrace{\alpha}_{-21.3^\circ} + \underbrace{\frac{\pi}{4}}_{45^\circ} = 23.7^\circ$$



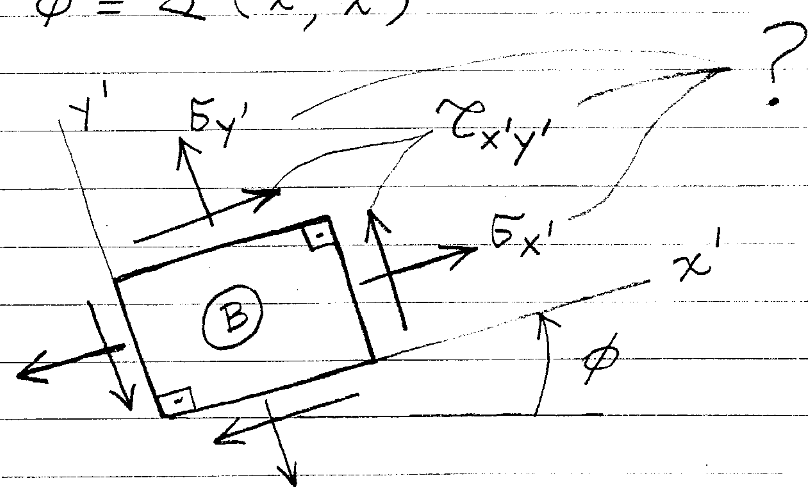
TRANSFORMATION FORMULAS



obtained from (A) by rotating it by the angle ϕ in the anticlockwise dir.



$$\phi = \measuredangle (x', x)$$



We wish to find $\sigma_{x'}$, $\sigma_{y'}$, $\tau_{x'y'}$ in terms of known σ_x, σ_y, τ_{xy}

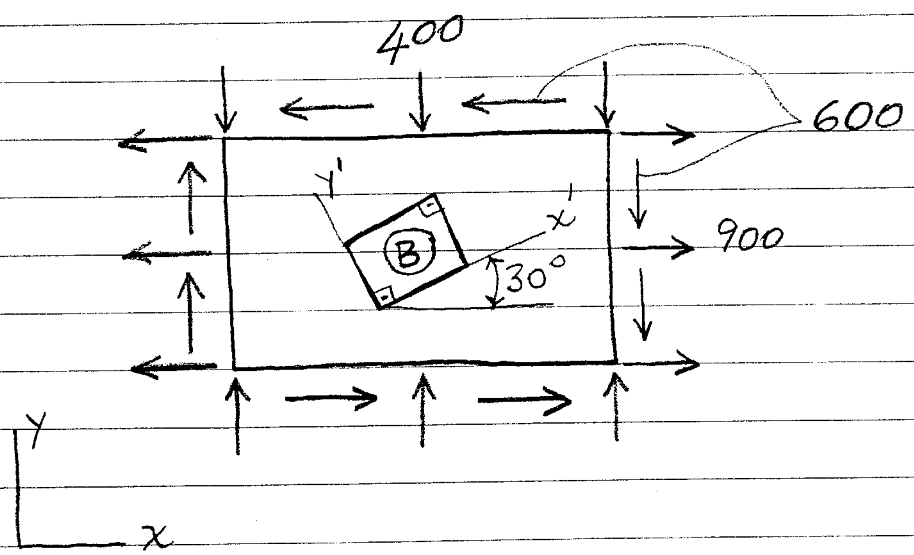
They can be found by using the following transformation formulas:

$$\text{①}\quad \sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\phi + \tau_{xy} \sin 2\phi$$

$$\text{②}\quad \sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\phi - \tau_{xy} \sin 2\phi$$

$$\text{③}\quad \tau_{x'y'} = -\frac{(\sigma_x - \sigma_y)}{2} \sin 2\phi + \tau_{xy} \cos 2\phi$$

EXAMPLE: (Previous example)



we wish to find the stresses acting on B.

$$\phi = \measuredangle(x', x) = 30^\circ \Rightarrow 2\phi = 60^\circ$$

$$\Rightarrow \sin 2\phi = \frac{\sqrt{3}}{2}, \quad \cos 2\phi = \frac{1}{2}$$

$$\left.\begin{array}{l} \sigma_x = 900 \\ \sigma_y = -400 \\ \tau_{xy} = -600 \end{array}\right\} kgf/cm^2$$

$$\frac{\sigma_x + \sigma_y}{2} = 250, \quad \frac{\sigma_x - \sigma_y}{2} = 650$$

$$① \Rightarrow \sigma_{x'} = (250) + (650)(1/2) + (-600)(\sqrt{3}/2)$$

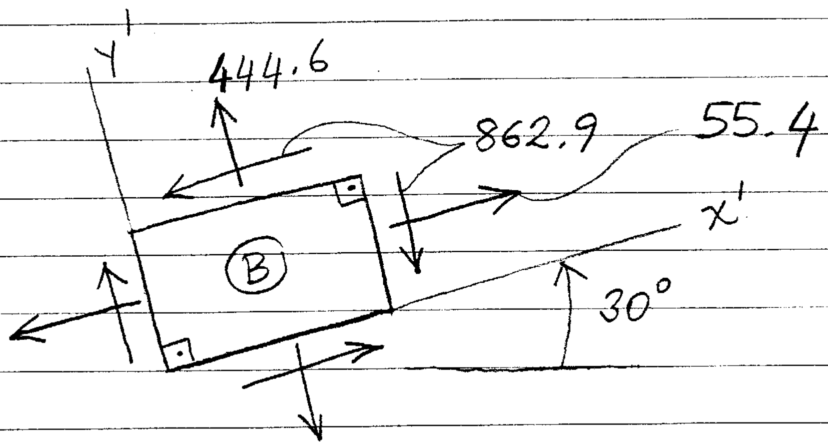
$$\Rightarrow \sigma_{x'} = 55.4 \text{ (T)}$$

$$② \Rightarrow \sigma_{y'} = (250) - (650)(1/2) - (-600)(\sqrt{3}/2)$$

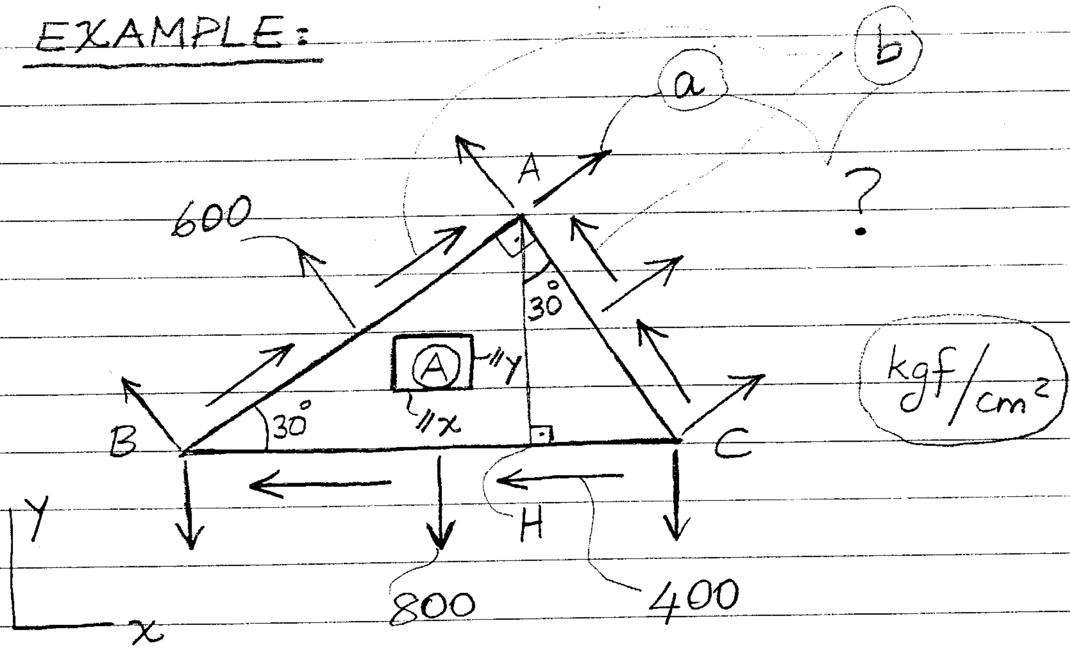
$$\Rightarrow \sigma_{y'} = 444.6 \text{ (T)}$$

$$③ \Rightarrow \tau_{x'y'} = -(650)(\sqrt{3}/2) + (-600)(1/2)$$

$$\Rightarrow \tau_{x'y'} = -862.9$$



EXAMPLE:



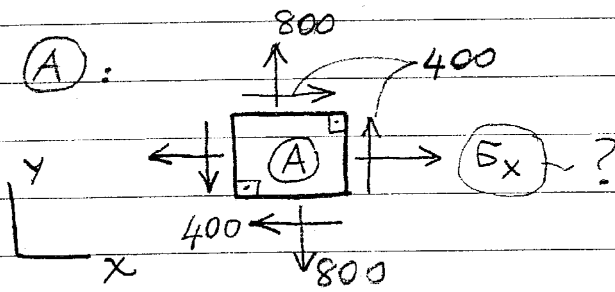
We wish to compute:

a) Stresses acting on (A) & the stresses "a" and "b"

b) PS's & PE

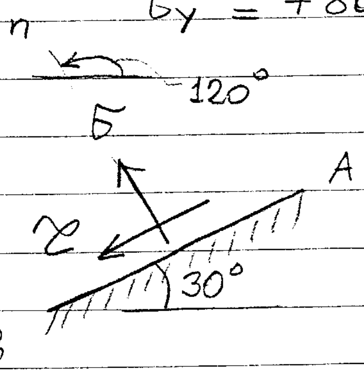
a) 1st method: Use formulas:

FBD of (A):



$\Rightarrow \sigma_x = ?$

$\sigma_y = +800, \quad \tau_{xy} = +400$



(**)

$\sigma = 600$

$\tau = -b$?

Write the formula for σ: ↘

$$\underbrace{\sigma}_{600} = \frac{\overset{?}{\sigma_x + \sigma_y}}{2} + \frac{\overset{?}{\sigma_x - \sigma_y}}{2}\cos 2\phi + \overset{400}{\tau_{xy}}\sin 2\phi \qquad (*)$$

800

$$\phi = 120^\circ \Rightarrow 2\phi = 240^\circ \Rightarrow \cos 2\phi = -1/2$$
$$\sin 2\phi = -\sqrt{3}/2$$

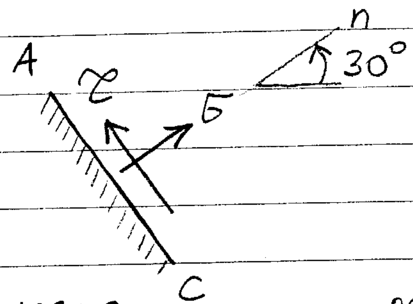
$$(*) \Rightarrow \boxed{\sigma_x \simeq 1385.6 \ \text{kgf/cm}^2 \ (T)}$$

a, b ?

$$(**) \Rightarrow \underbrace{\tau}_{-b} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\phi + \tau_{xy}\cos 2\phi$$

$$\Rightarrow \boxed{b = -53.6}$$

a?



$\sigma = a$
$\tau = b$

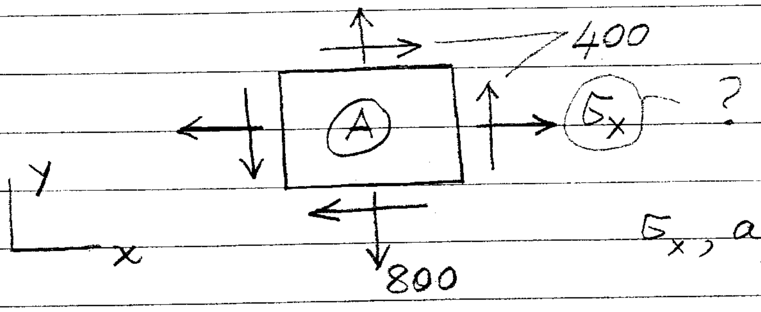
$$\underbrace{\sigma}_{a} = \overset{1092.8}{\frac{\sigma_x + \sigma_y}{2}} + \overset{292.8}{\frac{\sigma_x - \sigma_y}{2}}\cos 2\phi + \overset{400}{\tau_{xy}}\sin 2\phi$$

$$\phi = 30^\circ \Rightarrow 2\phi = 60^\circ$$

$$\Rightarrow \cos 2\phi = 1/2$$
$$\sin 2\phi = \sqrt{3}/2$$

$$\Rightarrow \boxed{a = 1585.6 \ \text{kgf/cm}^2}$$

2nd method: Use EE:



σ_x, a, b ?

a, b ?

Consider the triangle plate:

$$\overset{+}{\nwarrow} \sum F_{AC} = 0$$

$$\Rightarrow b * \overline{AC} * \cancel{t} + 600 * \overline{AB} * \cancel{t} + 400 * \cos 60^\circ * \overline{BC} * \cancel{t} - 800 * \cos 30^\circ * \overline{BC} * \cancel{t} = 0$$

$$\Rightarrow b * \underbrace{\frac{\overline{AC}}{\overline{BC}}}_{\sin 30^\circ} + 600 * \underbrace{\frac{\overline{AB}}{\overline{BC}}}_{\cos 30^\circ} + 400 * \frac{1}{2} - 800 * \frac{\sqrt{3}}{2} = 0$$

$$\Rightarrow \boxed{b = -53.6 \ kgf/cm^2}$$

the same as the one found previously

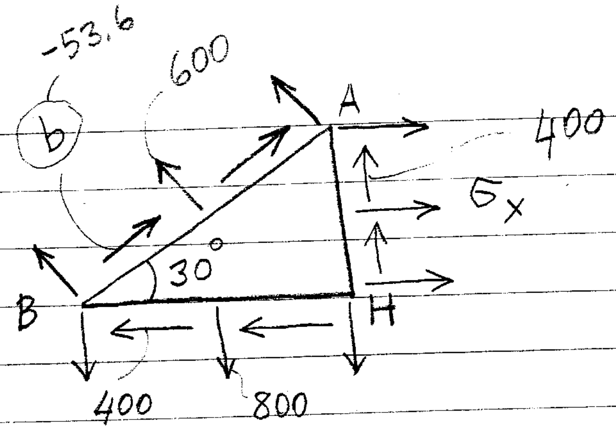
a ?

$$\overset{+}{\nearrow} \sum F_{AB} = 0$$

$$\Rightarrow \boxed{a = 1585.6}$$

$\sigma_x = ?$ by EE

$\longrightarrow$

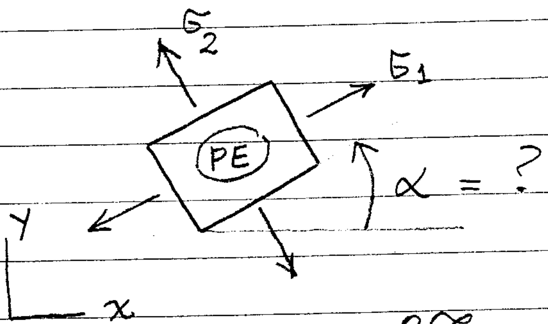


$\sum F_x = 0$

$\Rightarrow \boxed{\sigma_x = 1385.6}$

b) PS's & PE ?

$$\left.\begin{aligned} \sigma_x &= 1385.6 \\ \sigma_y &= 800 \\ \tau_{xy} &= 400 \end{aligned}\right\} \Rightarrow \begin{aligned} \frac{\sigma_x+\sigma_y}{2} &= 1092.8 \\ \frac{\sigma_x-\sigma_y}{2} &= 292.8 \end{aligned}$$



$$\tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x-\sigma_y} = 1.366 \Rightarrow 2\alpha \cong 53.8° \cancel{+\pi} \ ?$$

$$\sin 2\alpha = \frac{\tau_{xy}}{r} > 0$$

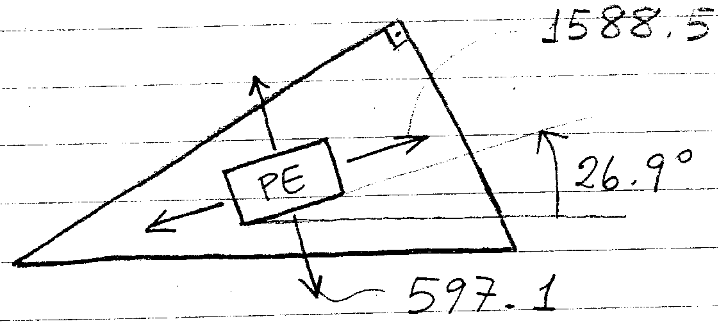
$\Rightarrow \boxed{\alpha = 26.9°}$

$$r = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2} = 495.7 \text{ kgf/cm}^2$$

$$\sigma_1 = \frac{\sigma_x+\sigma_y}{2} + r = 1588.5 \ (T)$$

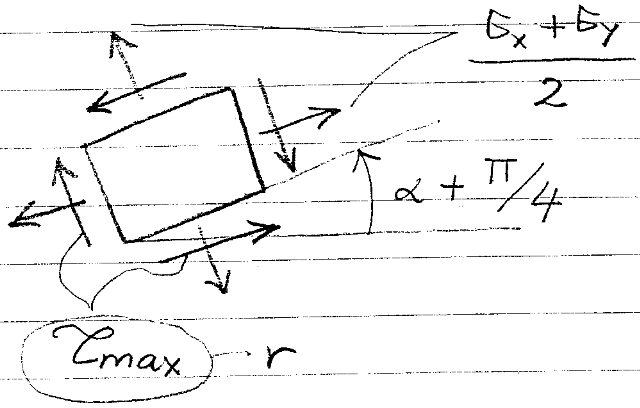
1092.8

$$\sigma_2 = \frac{\sigma_x+\sigma_y}{2} - r = 597.1 \ (T)$$



τ_{max} & the element on which it acts?

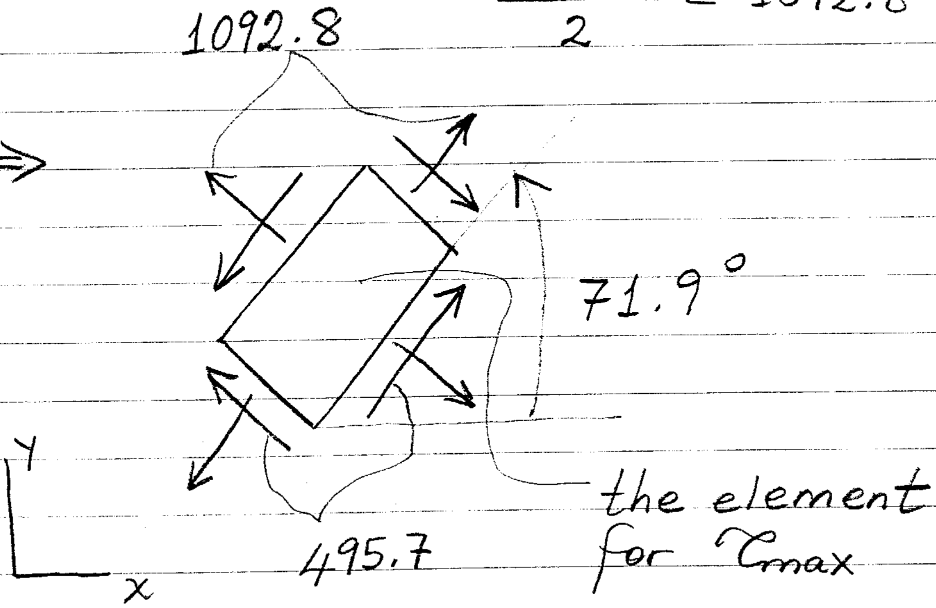
$$\boxed{\tau_{max} = r = 495.7 \ kgf/cm^2}$$



In our problem: $\alpha = 26.9° \Rightarrow \alpha + \frac{\pi}{4} = 71.9°$

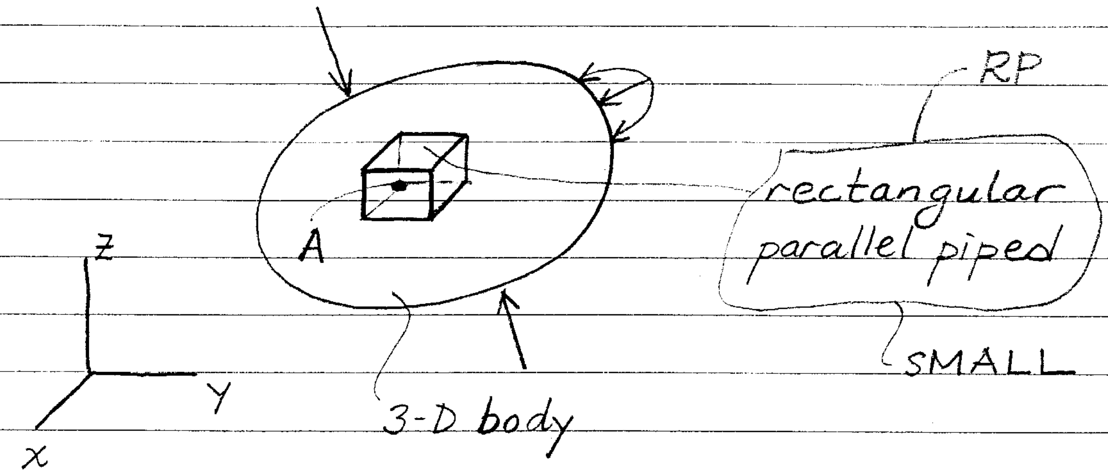
$$\frac{\sigma_x + \sigma_y}{2} = 1092.8$$

$\Rightarrow$

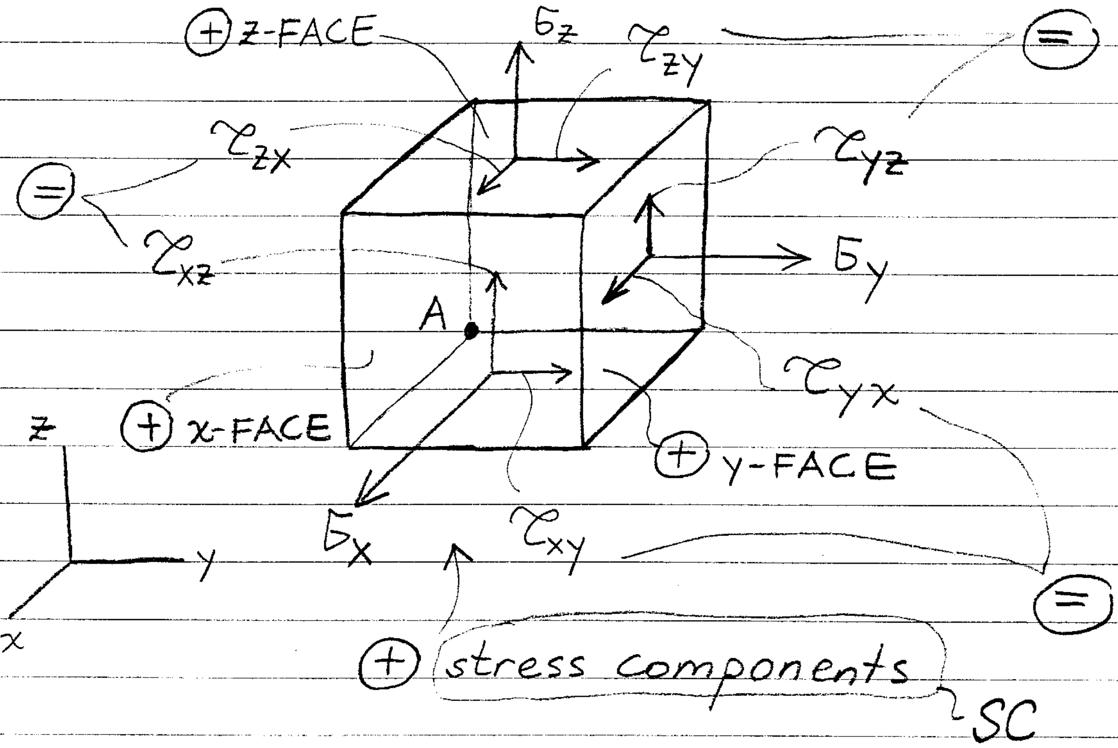


THREE DIMENSIONAL STATE OF STRESS

3-D



FBD of RP:



On ⊖ faces, SC's will act in ⊖ dir.

$$\begin{pmatrix} \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{pmatrix} \rightarrow SS\text{'s} \quad ; \quad \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} \rightarrow NS\text{'s} \begin{cases} \oplus \text{ if tensile} \\ \ominus \text{ if compressional} \end{cases}$$

Matrix representation of SC's:

$$\underbrace{\underline{\sigma}}_{\text{stress matrix}} = \begin{array}{c} \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \text{SC's on } x\text{-face} \\ \leftarrow \text{ " } \quad \text{ " } \quad y\text{- "} \\ \leftarrow \text{ " } \quad \text{ " } \quad z\text{- "} \end{array}$$

$=$ $=$ $=$

NS's

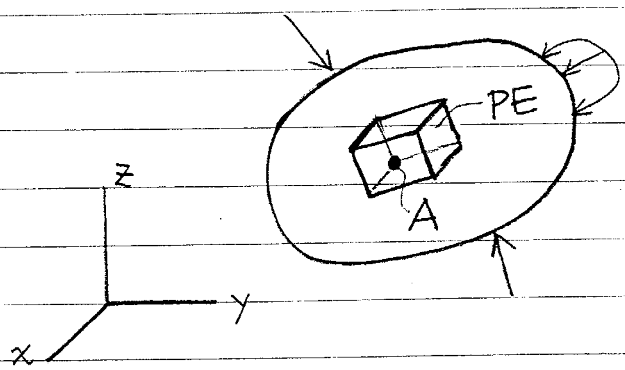
$$\Rightarrow \quad \underline{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

SS's $=$

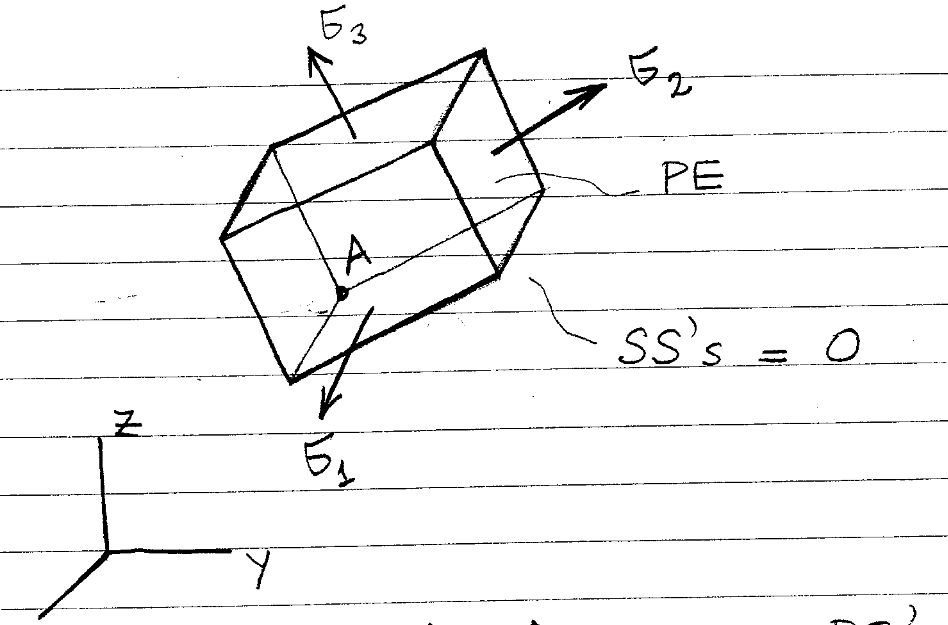
(3x3) symmetric stress matrix

contains 6 independent SC's $\rightarrow \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} \;\&\; \begin{pmatrix} \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{pmatrix}$

PS's & PE FOR 3-D CASE



PE at the pt. A is a special RP on which only NS's act.

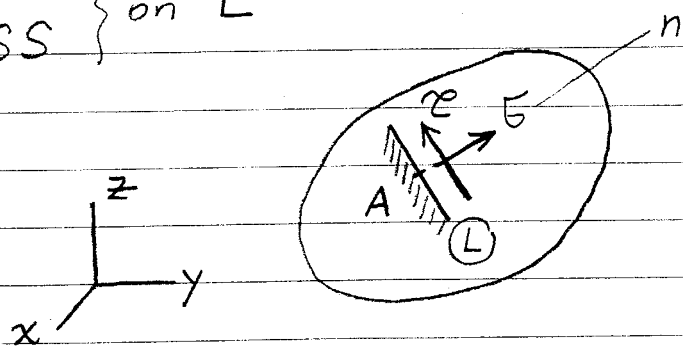


$\sigma_1, \sigma_2, \sigma_3 \rightarrow$ PS's

If $\underline{\sigma} = \begin{bmatrix} \sigma_x & \cdots \\ \vdots & \sigma_z \end{bmatrix}$ is given at the pt. A, PE and PS's can be determined by using some formulas. The establishment of these formulas is the subject of the theory of elasticity, and they will not be represented here.

NS & SS ACTING ON A FACE

$\left.\begin{array}{l} \sigma \rightarrow NS \\ \tau \rightarrow SS \end{array}\right\}$ on L



$\sigma \rightarrow +$ if tensile

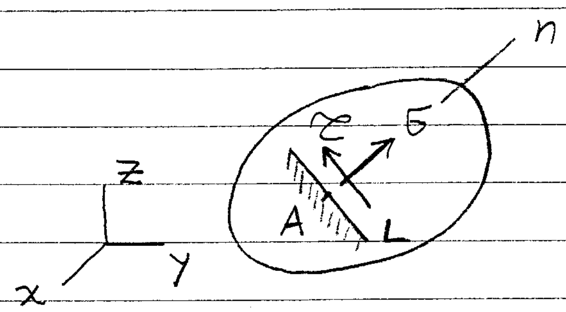
$\tau \rightarrow +$ if directed to the left.

If $\underline{\sigma} = \begin{bmatrix} \sigma_x & \cdots & \\ \vdots & & \\ & \cdots & \sigma_z \end{bmatrix}$ is given at "A", σ and τ values can be found by using some formulas.

They will not be given in here.

As L changes: σ & τ will also change.

MC FOR 3-D CASE



Suppose that $\underline{\sigma} = \begin{bmatrix} \sigma_x & \cdots & \\ \vdots & & \\ & \cdots & \sigma_z \end{bmatrix}$ is given at "A"

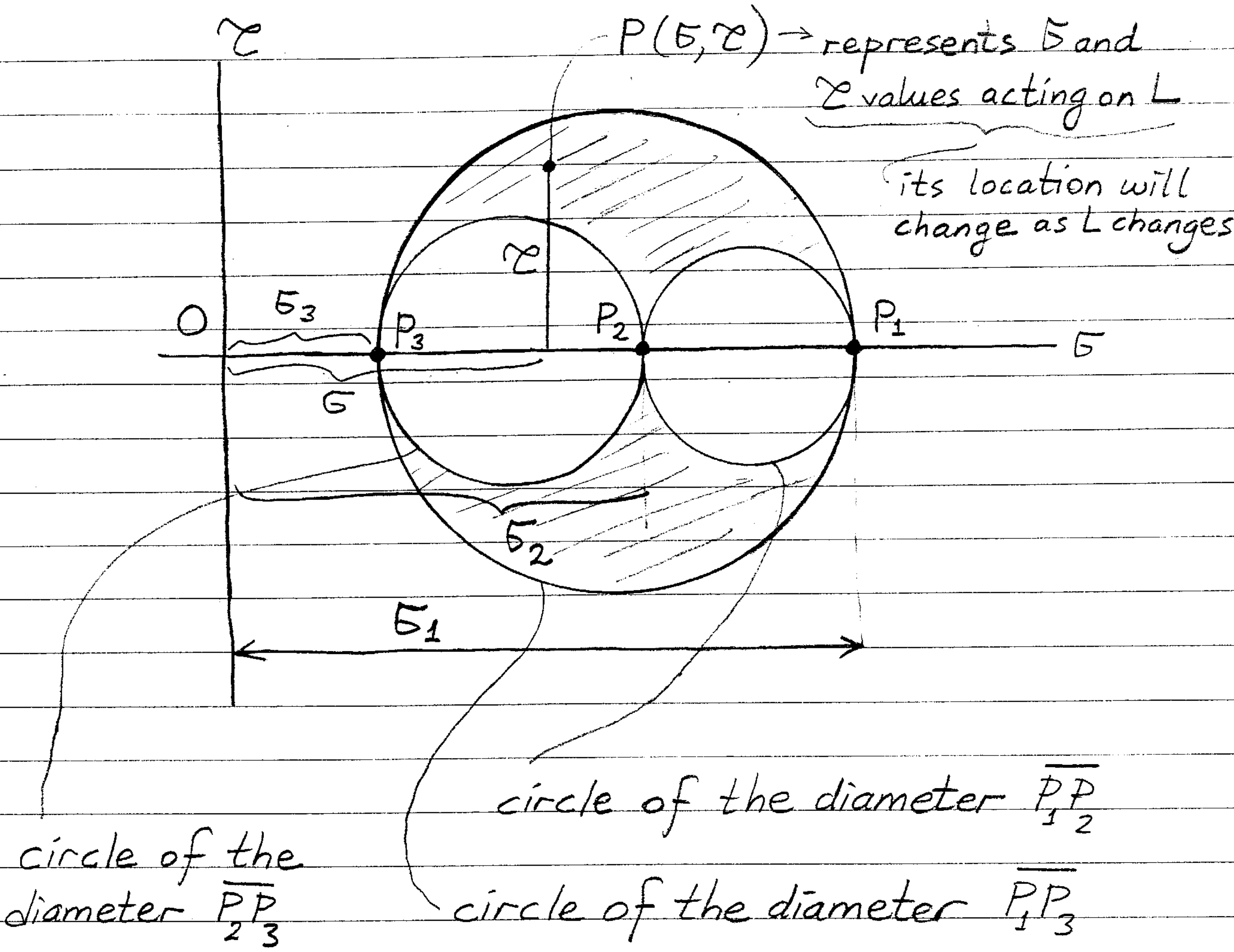
Thus, we can find PE & $\underbrace{\text{PS's}}_{\sigma_1,\ \sigma_2,\ \sigma_3}$ at A.

Knowing PS's we can construct 3-D MC as follows:

1) Order σ_1, σ_2, σ_3 so that $\underbrace{\sigma_1 > \sigma_2 > \sigma_3}_{\text{including the SIGN}}$

Ex: PS's $\rightarrow$ $(\underbrace{-10}_{\sigma_2}, \underbrace{-20}_{\sigma_3}, \underbrace{5}_{\sigma_1})$

2) Draw 3 circles :

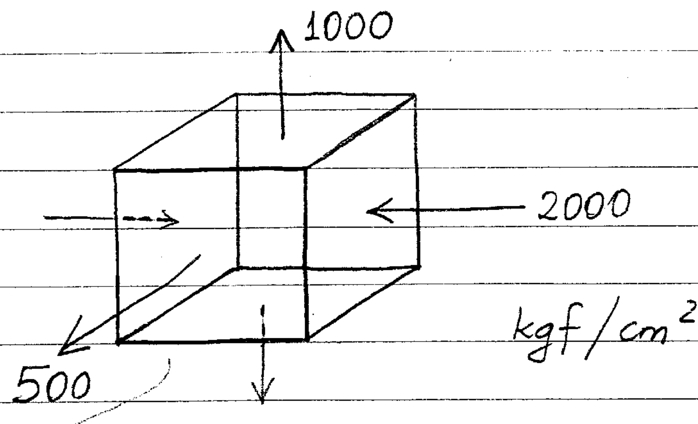


It can be shown that for all faces L the point P remains in the shaded region. These three circles are called MC for 3-D case.

$$\tau_{max} = \max|\tau| = \text{radius of outer MC}$$

$$\Longrightarrow \boxed{\tau_{max} = \frac{\sigma_1 - \sigma_3}{2}}$$

EXAMPLE:

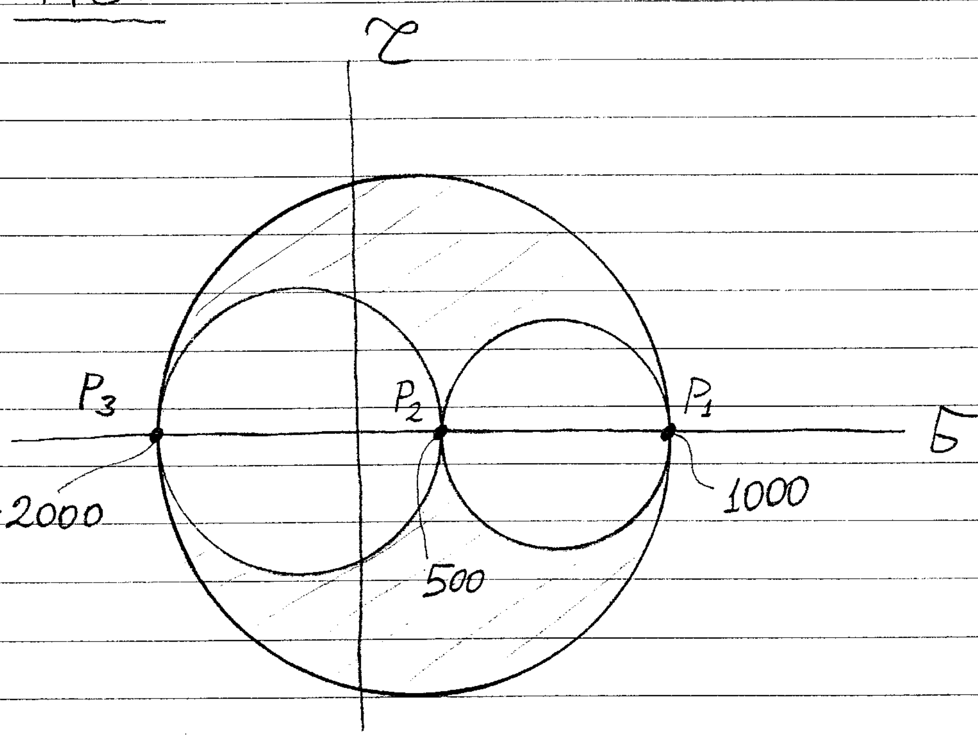


subjected only to NS's $\Rightarrow$ PE

$$\Rightarrow PS's \rightarrow (\underbrace{500}_{\sigma_2}, \underbrace{-2000}_{\sigma_3}, \underbrace{1000}_{\sigma_1})$$

$$\Rightarrow \sigma_1 > \sigma_2 > \sigma_3$$

MC:



$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{1000 - (-2000)}{2} = 1500 \ kgf/cm^2$$