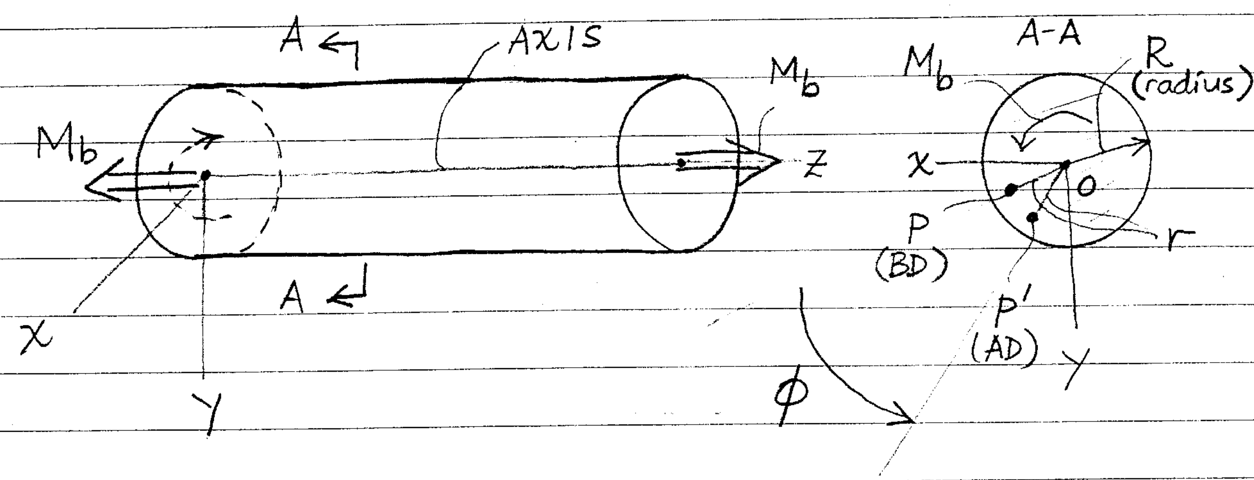
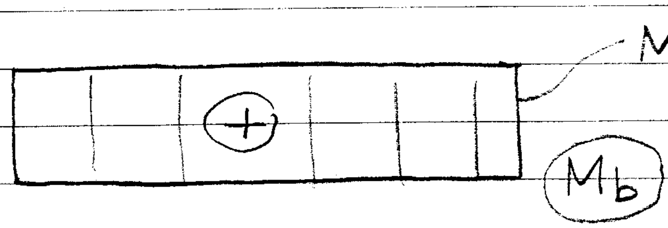
CHAPTER 6

TORSION OF CYLINDRICAL BARS

Assume that cross section (CS) of the bar is circular.





$M_b \rightarrow$ twisting moment (TM)

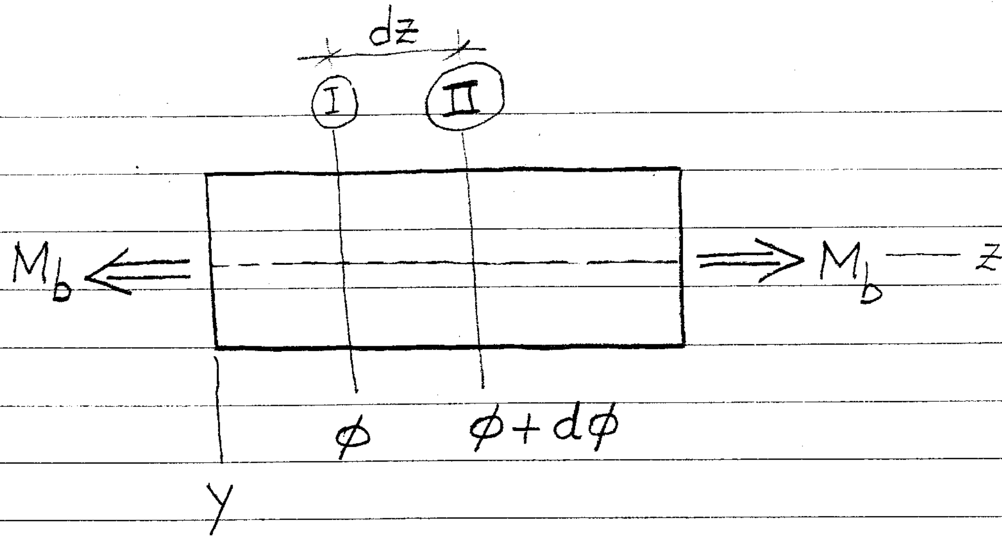
$\phi \rightarrow$ angle of rotation

describes the rotation of CS

$O \rightarrow$ center of rotation (CR)

$\overline{OP'} = \overline{OP} = r$

$\phi \rightarrow$ changes from one CS to another



The relative rotation of II wrt I is:

$$\phi + d\phi - \phi = d\phi$$

Relative rotation per unit length:

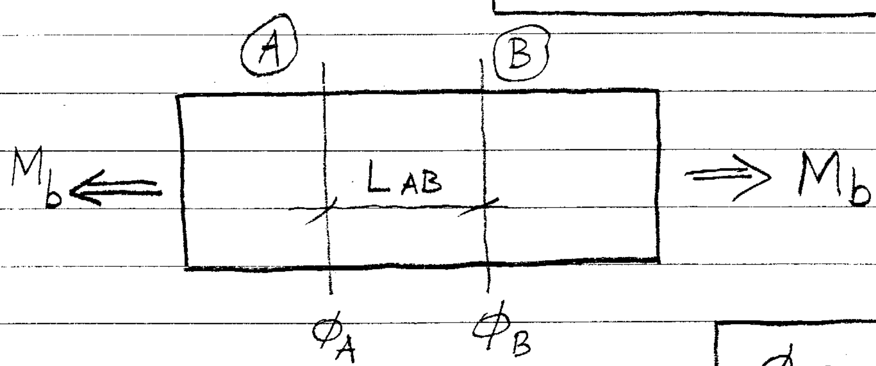
$$\boxed{\frac{d\phi}{dz} = \omega}$$

TA twist angle per unit length

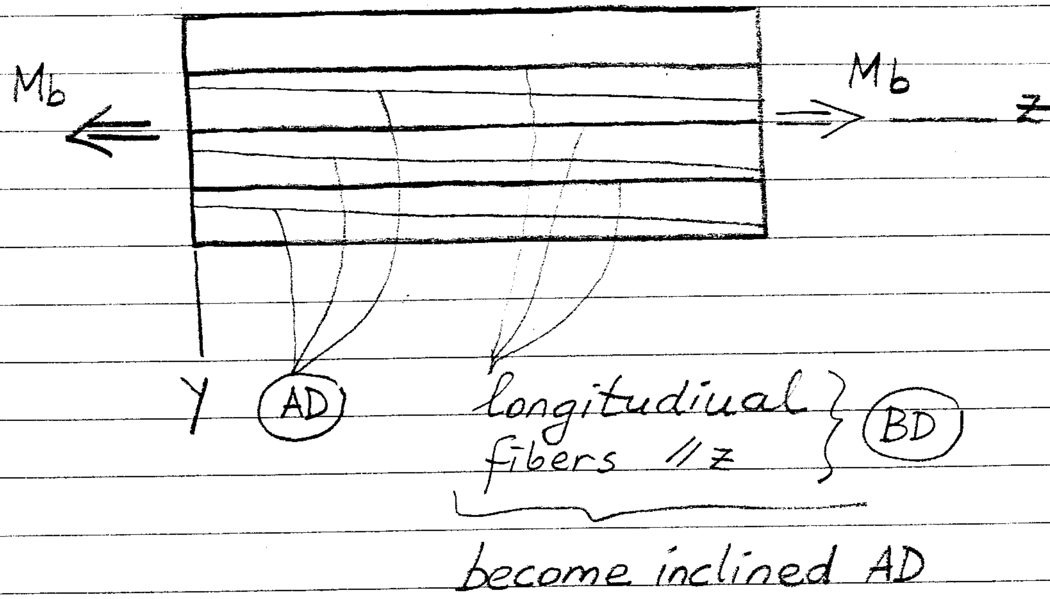
$[\omega] \rightarrow rad/L \rightarrow rad/cm, rad/m, ..$

$|\omega| \rightarrow$ small

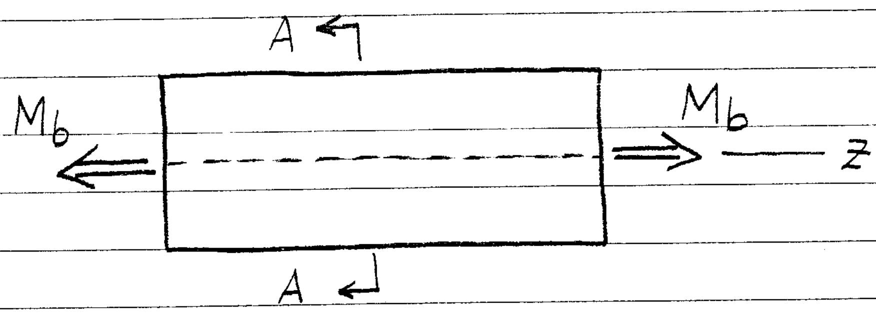
Assume : $\boxed{\omega = const.}$

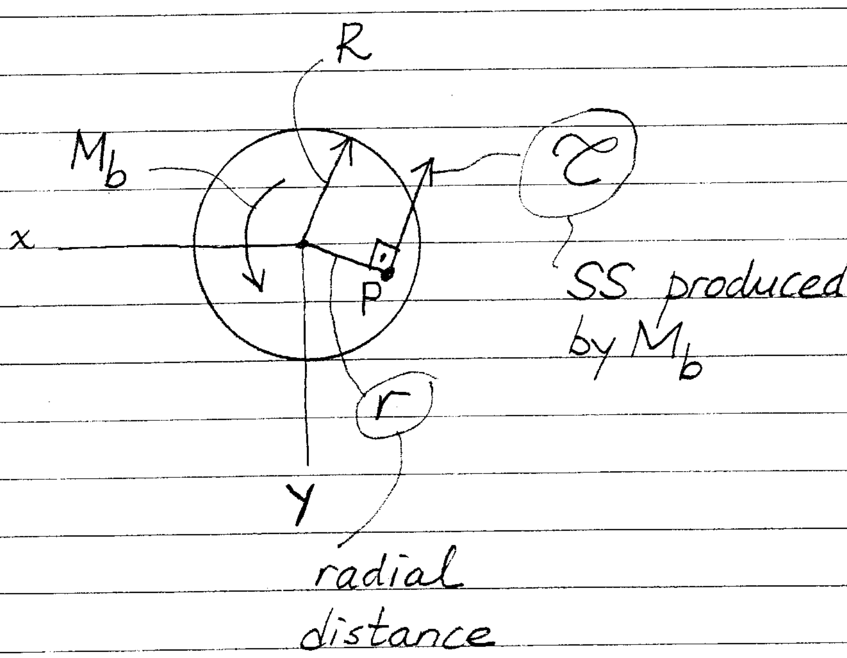


$$\boxed{\phi_B - \phi_A = \omega * L_{AB}}$$



STRESS PRODUCED BY M_b :



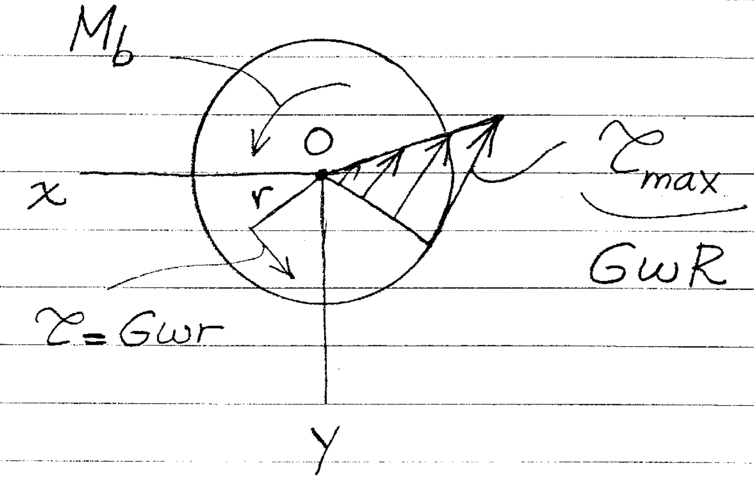


It can be shown that:

$$\boxed{\tau = \underbrace{G}_{\text{shear modulus}}\ \underbrace{\omega}_{TA}\ \underbrace{r}_{\text{radial distance}}}$$

depends linearly on "r"

$$\tau \begin{cases} \rightarrow \text{at } r = 0 \quad \tau = 0 \\ \rightarrow \text{at } r = R \quad \tau = \boxed{G\omega R = \tau_{max}} \end{cases}$$



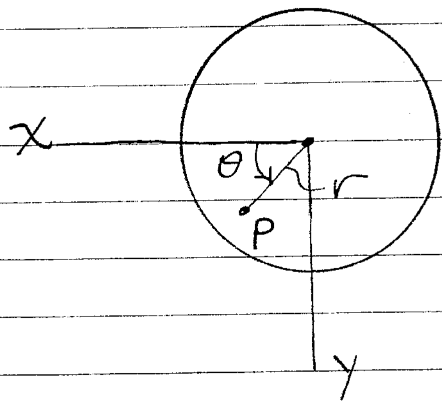
$$\boxed{\tau = G\omega r}$$

related to ω ?

wish to relate τ to M_b

Preparations:

$\overbrace{\text{polar coordinate system}}^{\text{PCS}}$

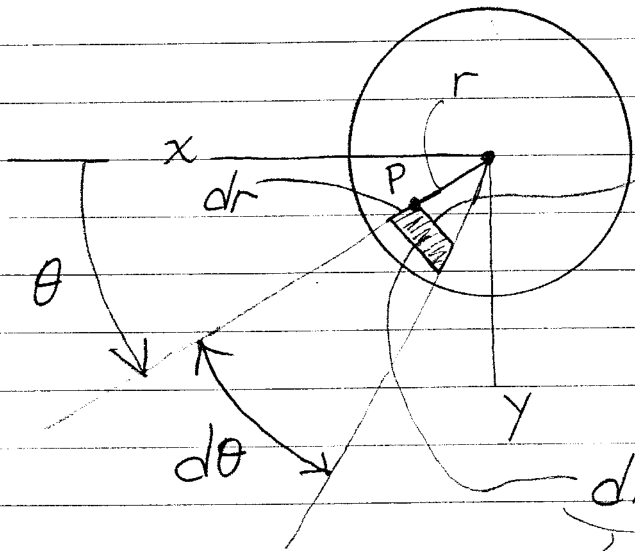


$r, \theta \rightarrow$ polar coordinates of P

angle measured from x-axis (θ)

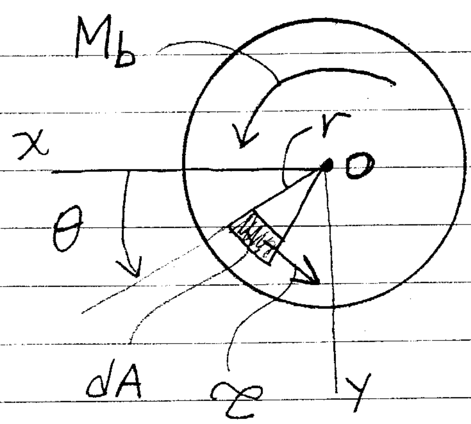
radial distance (r)

Small area in PCS:



$r\,d\theta$

$$\underbrace{dA}_{\text{small area in PCS}} = r\,d\theta\,dr$$



$$\int_{\underbrace{A}_{\text{area of CS}}} r * \overbrace{\tau}^{G\omega r} dA = M_b$$

$$\Rightarrow \int_A r * G\omega r \, dA = M_b$$

$$\Rightarrow G\omega \underbrace{\left(\int_A r^2 dA \right)}_{I_o} = M_b$$

polar inertia moment of CS

PIM

$$\Rightarrow G\omega * I_o = M_b$$

$$\Rightarrow \boxed{\omega = \frac{M_b}{G I_o}} \quad (*)$$

determines ω in terms of M_b

$GI_o \rightarrow$ torsional rigidity of CS

As $GI_o \rightarrow \infty \Rightarrow \omega \rightarrow 0$

Computation of I_o:

$$I_o = \int_A r^2 dA = \iint r^2 (r \, dr \, d\theta)$$

$$= \int_0^{2\pi} \left[\int_0^R r^3 dr \right] d\theta$$

$$\Rightarrow \quad I_o = \int_0^{2\pi} \frac{R^4}{4}\, d\theta = \frac{R^4}{\cancel{4}_{\,2}} \underbrace{\int_0^{2\pi} d\theta}_{2\pi}$$

$$\Rightarrow \quad \boxed{I_o = \frac{\pi R^4}{2}}$$ PIM for circular CS

$$[I_o] \rightarrow L^4 \Rightarrow cm^4, m^4, \ldots$$

Recall:

$$\tau = G\omega r \quad (1)$$

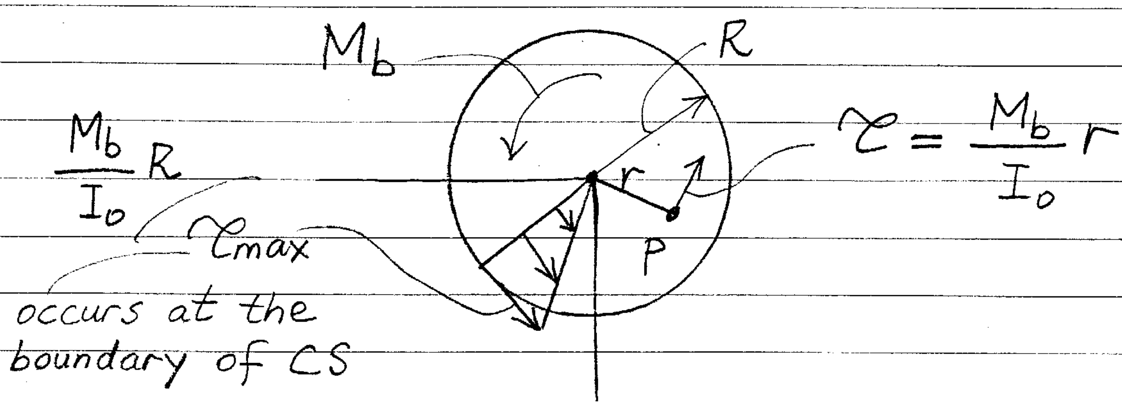
$$(*) \Rightarrow \quad \omega = \frac{M_b}{G I_o} \quad (2)$$

$(2) \rightarrow (1)$:

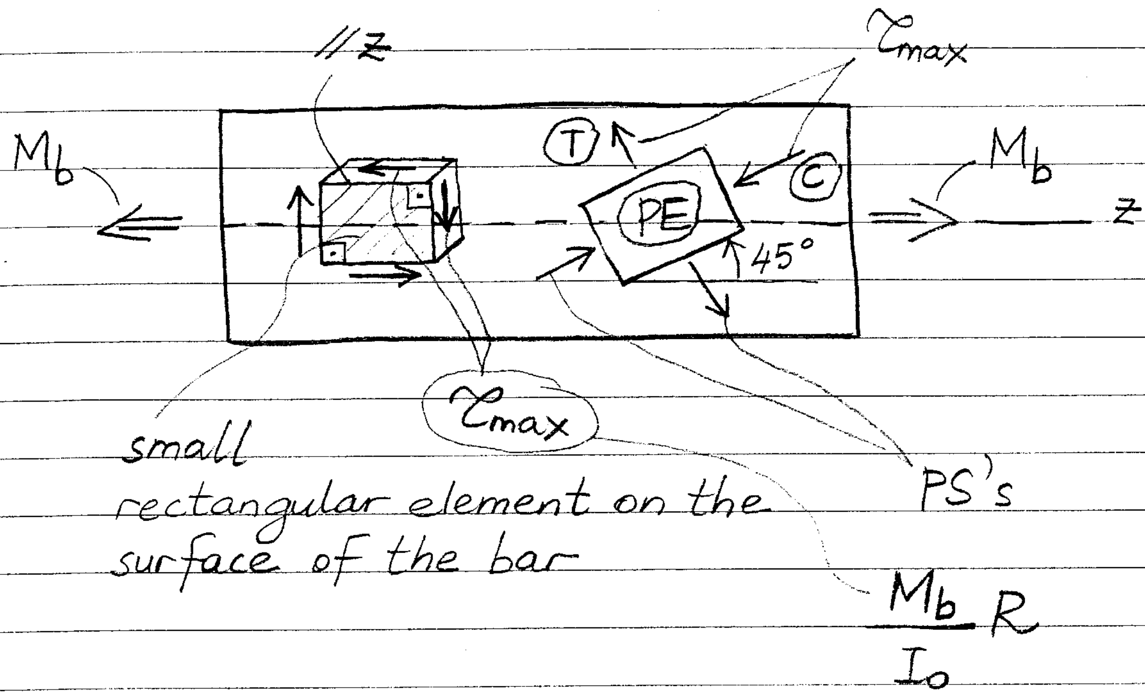
$$\Rightarrow \quad \tau = \cancel{G} * \frac{M_b}{\cancel{G} I_o} r$$

$$\Rightarrow \quad \boxed{\tau = \frac{M_b}{I_o} r}$$ depends linearly on r

determines τ in terms of M_b

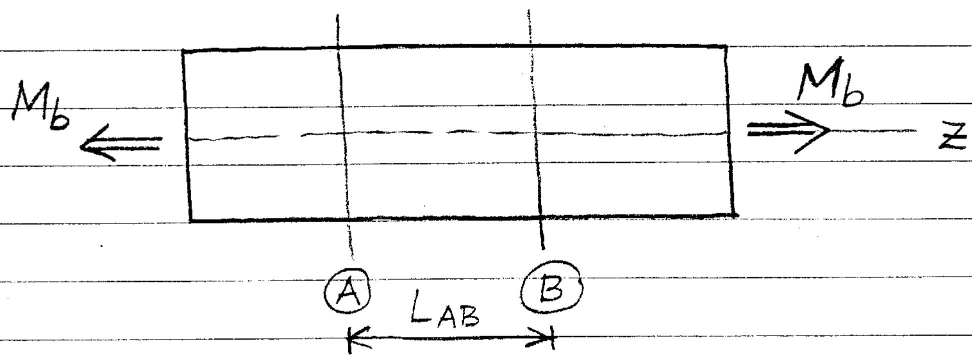


PS's CAUSED BY THE TORSION



In torsion, PE makes an angle of 45° with z-axis.

COMPUTATION OF THE ANGLE OF ROTATION

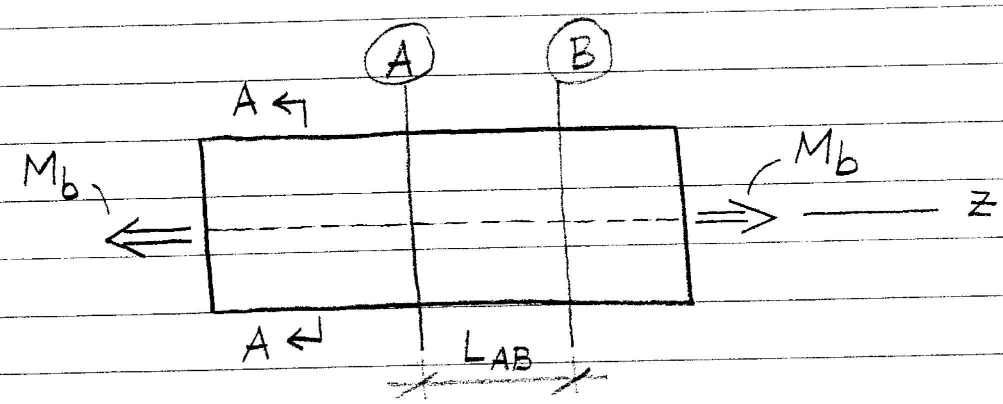


$$\phi_B - \phi_A = \underbrace{\omega}_{\frac{M_b}{G I_o}} * L_{AB}$$

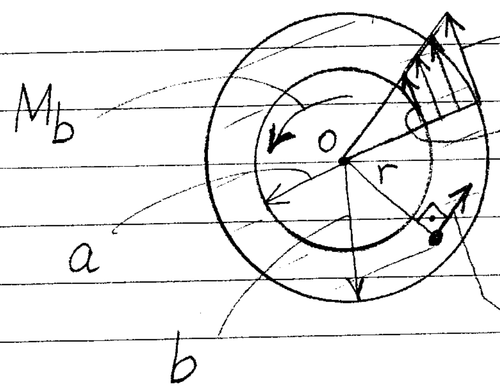
$$\Rightarrow \quad \boxed{\phi_B - \phi_A = \frac{M_b L_{AB}}{G I_o}}$$

determines relative rotation of B wrt A in terms of M_b.

TORSION OF HOLLOW CIRCULAR BARS



A-A



$$\tau_{max} = \frac{M_b}{I_o} * b$$

$$\tau_{min} = \frac{M_b}{I_o} * a$$

$$\tau = \frac{M_b}{I_o} r$$

$$\omega = \frac{M_b}{G I_o}$$

$$I_o = \frac{\pi}{2}(b^4 - a^4)$$

$$\phi_B - \phi_A = \frac{M_b L_{AB}}{G I_o}$$

DESIGN EQUATION FOR TORSION

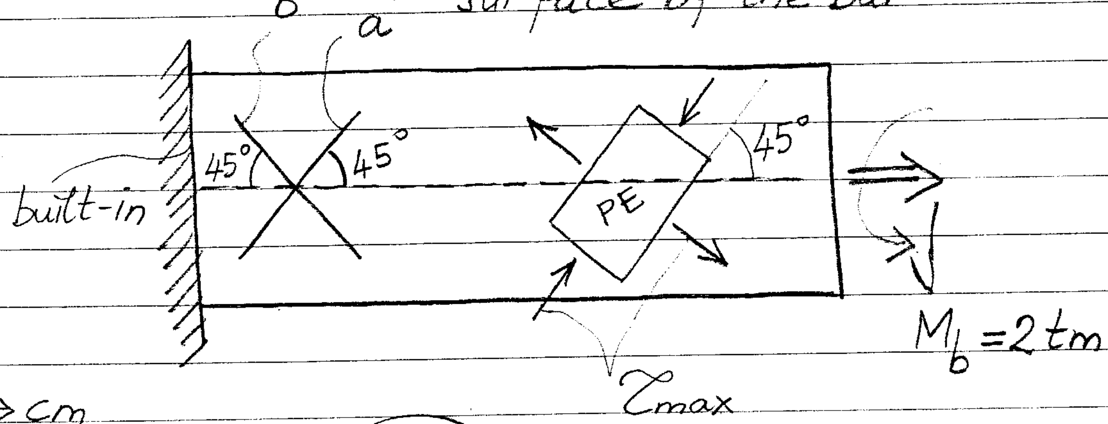
$$\boxed{\tau_{max} \leq \tau_a}$$

τ_a: allowable SS for torsion

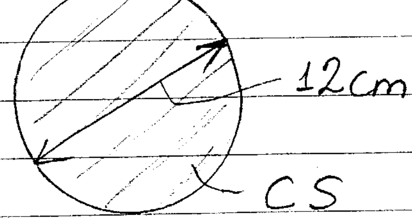
$\tau_{max} = \dfrac{M_b}{I_o} * R$ for solid circular CS, $I_o = \pi R^4/2$

$\tau_{max} = \dfrac{M_b}{I_o} * b$ for hollow circular CS, $I_o = \dfrac{\pi}{2}(b^4 - a^4)$

EXAMPLE:



$L \to cm$
$F \to kgf$



$\left(G = \dfrac{E}{2(1+\nu)} \right)$

$\left.\begin{aligned} \varepsilon_a &= -4 * 10^{-4} \\ \varepsilon_b &= 4 * 10^{-4} \end{aligned}\right\}$ cm/cm measured

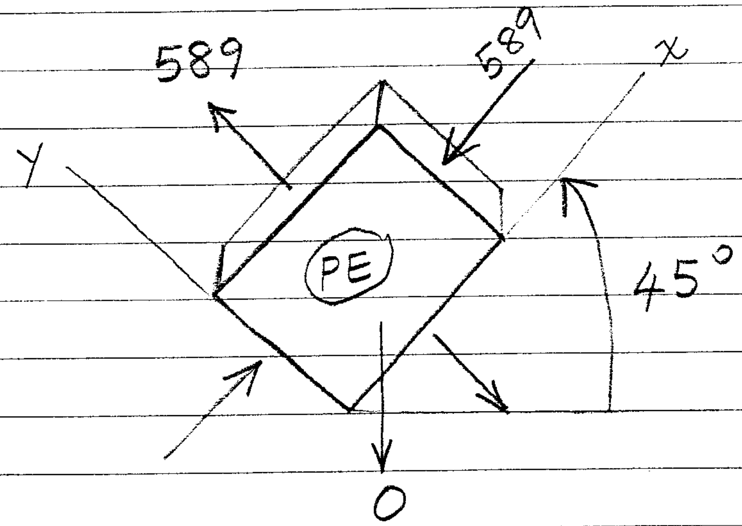
$G = ?$ shear modulus

$$\tau_{max} = \frac{M_b}{I_o} * R$$

$$M_b = 2\,tm = 2*10^5\ kgf\text{-}cm$$

$$I_o = \frac{\pi R^4}{2} = \frac{\pi (6)^4}{2} \cong 2036\ cm^4$$

$$\Rightarrow \boxed{\tau_{max} \cong 589\ kgf/cm^2}$$



$$\Rightarrow \sigma_x = -589\ ,\quad \sigma_y = +589$$

HES for ε_x :

$$\underbrace{\varepsilon_x}_{\varepsilon_a = -4*10^{-4}} = \frac{1}{E}\left(\overset{-589}{\sigma_x} - \nu \overset{589}{\sigma_y} - \nu \overset{0}{\cancel{\sigma_z}}\right)$$

$$\Rightarrow -4*10^{-4} = \frac{1}{E}\left(-589 - \nu * 589\right)$$

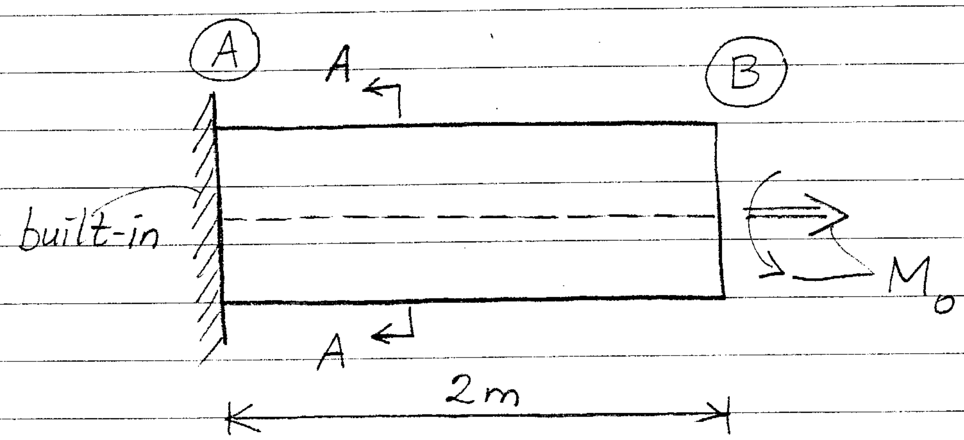
$$= -\frac{(1+\nu)}{E} * 589$$

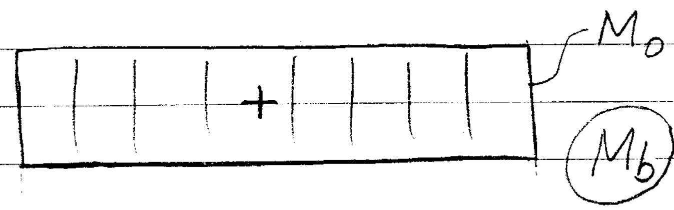
$$\Rightarrow \frac{E}{(1+\nu)} = \frac{589}{4 * 10^{-4}}$$

$$G = \frac{E}{2(1+\nu)} \Rightarrow \frac{E}{1+\nu} = 2G$$

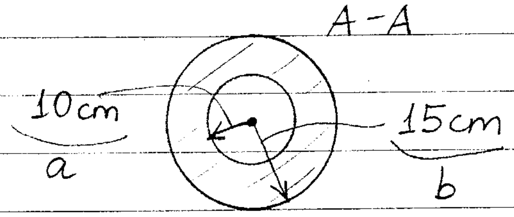
$$\Rightarrow \boxed{G = \frac{589}{8} * 10^{4} \cong 0.74 * 10^{6}\ kgf/cm^{2}}$$

EXAMPLE:





A-A



F → kgf
L → cm

$$\tau_a = 600\ kgf/cm^{2}$$

$$G = 0.8 * 10^{6}\ kgf/cm^{2}$$

a) max M_o which the bar can carry safely ?

b) $\phi_B = ?$

a) $$\tau_{max} = \frac{M_o}{\underbrace{I_o}_{?}} * \overbrace{b}^{15} \leq \overbrace{\tau_a}^{600} \quad \circledast$$

$$I_o = \frac{\pi}{2}(b^4 - a^4) = \frac{\pi}{2}(15^4 - 10^4)$$

$$= 63813 \text{ cm}^4$$

$$\circledast \Rightarrow M_o \leq \underbrace{2.55 * 10^6}_{25.5 \text{ ton-m}} \text{ kgf-cm}$$

$$\Rightarrow \boxed{(M_o)_{max} = 25.5 \text{ t-m}}$$

b) $$\phi_B - \overbrace{\phi_A}^{0} = \frac{M_b * \overbrace{L_{AB}}^{200\text{cm}}}{G\, I_o}$$

M_b: $(M_o)_{max} = 2.55 \times 10^6$

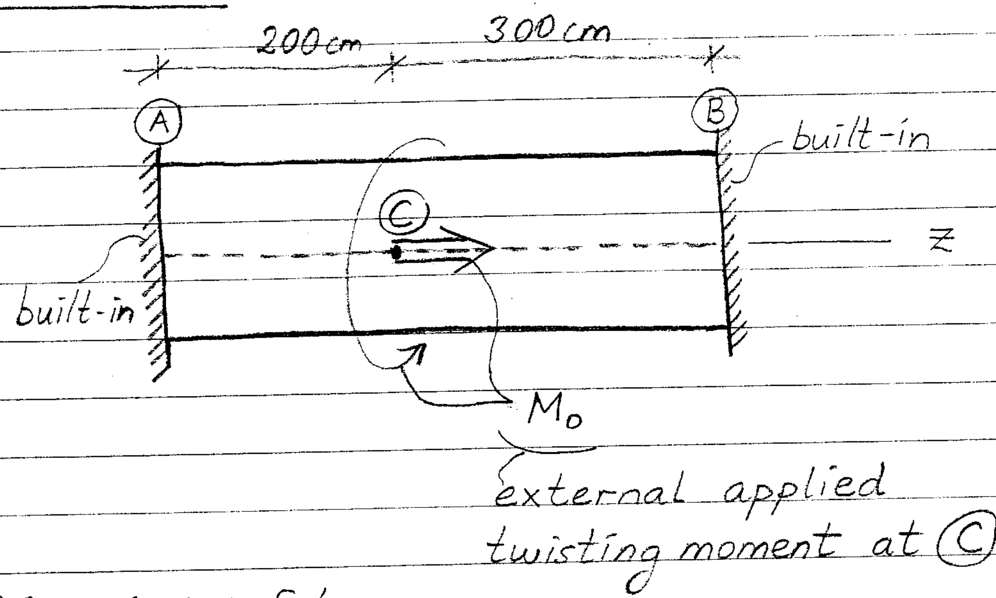
I_o: 63813

G: $0.8 * 10^6$

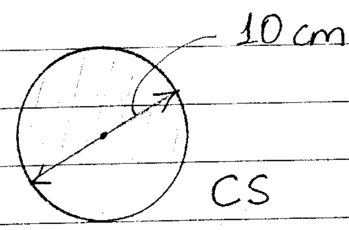
$$\Rightarrow \boxed{\phi_B \cong 0.008 \text{ rad.}}$$

VERY SMALL

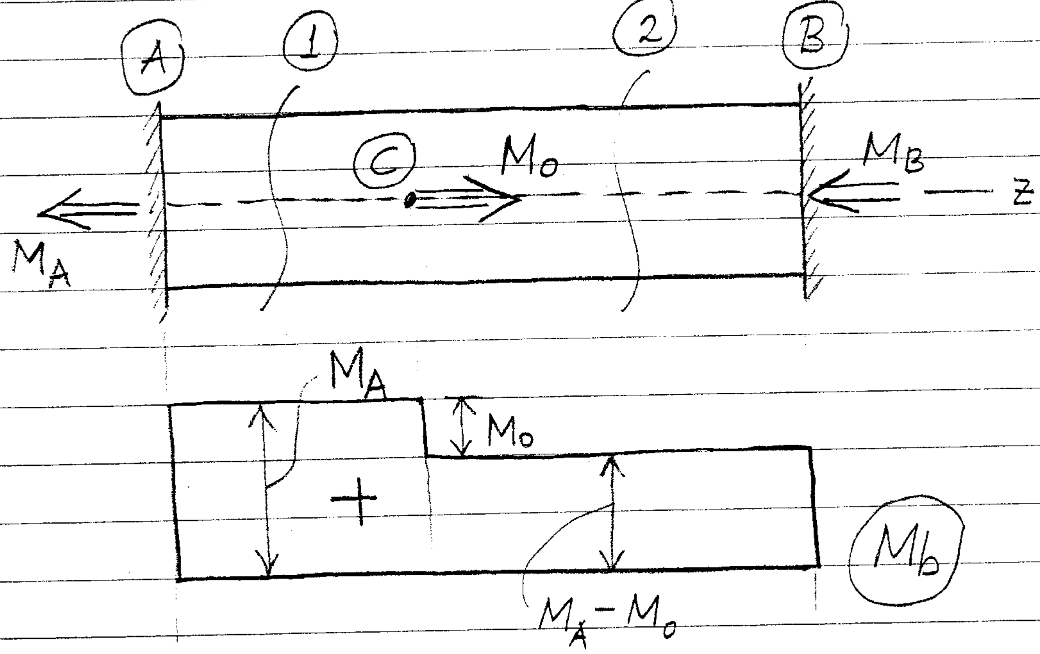
EXAMPLE :



$\tau_a = 600 \ kgf/cm^2$



wish to find max M_o which the bar can carry safely



$M_A, M_B \rightarrow$ support moments

$$\overset{+}{\rightarrow} \sum M_z = 0$$

$$\Rightarrow \quad M_o - M_A - M_B = 0$$

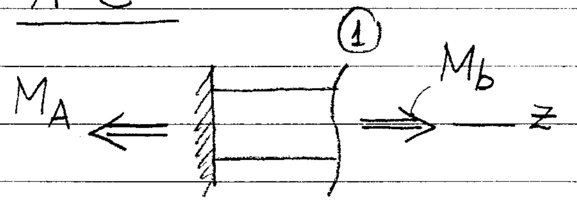
$$\Rightarrow \quad \boxed{M_A + M_B = M_o} \quad (1)$$

AE	U
$(1) \rightarrow 1$	$\left.\begin{matrix} M_A \\ M_B \end{matrix}\right) 2$

$$\Rightarrow \quad 2 - 1 = 1$$

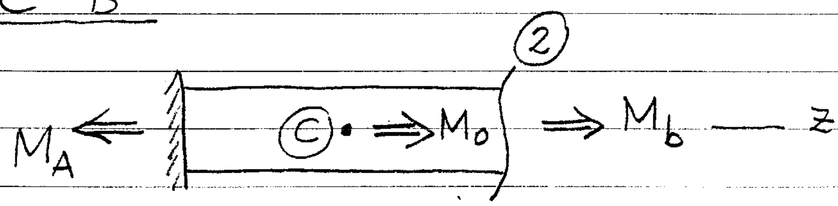
# of eqs. which we need

A-C



$$\Rightarrow \quad M_b = M_A$$

C-B



$$\overset{+}{\rightarrow} \sum M_z = 0$$

$$\Rightarrow \quad M_b + M_o - M_A = 0$$

$$\Rightarrow \quad \boxed{M_b = M_A - M_o}$$

Compatibility Equation: (CE)

$$\phi_B - \phi_A = 0$$

$$\Rightarrow (\phi_C - \phi_A) + (\phi_B - \phi_C) = 0$$

$$\Rightarrow \boxed{\frac{M_A * 200}{\cancel{GI_o}} + \frac{(M_A - M_o) * 300}{\cancel{GI_o}} = 0}$$

CE

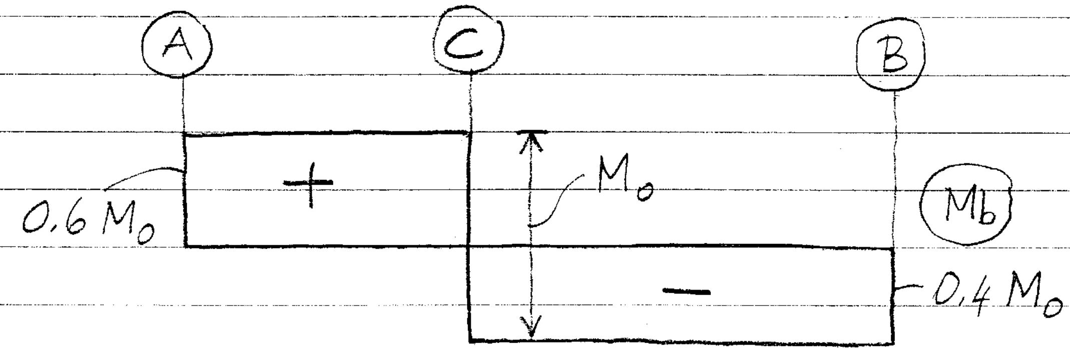
$$\Rightarrow 2M_A + 3(M_A - M_o) = 0$$

$$\Rightarrow 5M_A = 3M_o$$

$$\Rightarrow \boxed{M_A = 0.6\,M_o}$$

$$1 \Rightarrow \boxed{M_B = 0.4\,M_o}$$

$\Rightarrow$



max M_o = ?

$$\tau_{max} = \frac{\overbrace{M_b}^{(M_b)_{max} = 0.6M_o}}{\underbrace{I_o}_{\frac{\pi}{2}R^4}} \overbrace{R}^{5}$$

$$\Rightarrow \quad \frac{0.6\ M_o}{982} * 5 \leq 600$$

$$\Rightarrow \quad M_o \leq \underbrace{196400\ \text{kgf-cm}}_{\cong\ 1.96\ \text{ton-m}}$$

$$\Rightarrow \quad \boxed{\max M_o \cong 1.96\ \text{tm}}$$