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# YÜKSEK MERTEBEDEN DİFERENSİYEL DENKLEMLERİN ZAMAN SKALASINDA SALINIMLI VE SALINIMSIZLIK ÖZELLİKLERİ

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# TEŞEKKÜR

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# ÖZET

2. ve daha yüksek mertebeden lineer ve lineer olmayan diferensiyel ve dinamik denklemlerin çözümlerinin davranışı uzun yıllardır araştırma konusu olmuştur.

2. Mertebeden dinamik denklemler ve bu denklemlerin genelleştirilmiş versiyonları uzun yıllar bir çok araştırmaya konu olmuştur. Zaman Skalası teorisi ilk kez 1988 yılında Stefan Hilger tarafından doktora tezinde diskre analiz ve sürekliliği birleştirmek için ele alınmıştır.

Adi diferensiyel denklemler reel eksende keyfi kapalı alt cümlelerde çalışılmış denklemlerdir. Zaman skalası reel sayıların boş olmayan kapalı bir alt cümlesidir. $\mathbb{R}$; $\mathbb{Z}$; $\mathbb{N}$; $\mathbb{N}_0$; Yani reel sayılar , tam sayılar, doğal sayılar, negatif olmayan tamsayılar zaman skalası örnekleridir. Bununla birlikte rasyonel sayılar, irrasyonel sayılar, kompleks ve (0,1) açık aralığı zaman skalası değildir. Zaman skalası sembolü $\mathbb{T}$ ile gösterilir. Zaman skalasında dinamik denklemler popülasyon dinamikleri gibi konularda önemli uygulama alanlarına sahiptir. Örneğin bir böcek popülasyonunu bir mevsim boyunca sürekli bir model olarak alabiliriz. Kışın böcekler yumurtlar ve ölür ve yeni nesil yeni mevsimde yumurtadan çıkar ve eski ve yeni popülasyon üst üste gelmeden devam eder.

Biz bu çalışmada lineer ve lineer olmayan denklemler için verilmiş salınımlılık ve salınımsızlık kriterlerini genelleştirerek zaman skalasında ikinci mertebeden dinamik denklemlere uygulayacağız.

**Anahtar Kelimeler:** Zaman Skalası, Dinamik Denklem, Salınımlılık

## ABSTRACT

2. and higher-order linear and non-linear behavior of solutions of differential and dynamic equations has been the subject of research for many years.

2. Order dynamic equations and the generalized versions of these equations has been the subject of many studies for many years. Time Scale theory for the first time in his doctoral dissertation in 1988 by Stefan Hilger to combine discrete and continuous analysis are discussed. A time scale is an arbitrary nonempty closed subset of the real numbers. Thus $\mathbb{R}$; $\mathbb{Z}$; $\mathbb{N}$; $\mathbb{N}_0$; i.e., the real numbers, the integers, the natural numbers, and the nonnegative integers are examples of time scales. The rational numbers, the irrational numbers, the complex numbers, and the open interval between 0 and 1, are not time scales. Time scales denote by the symbol $\mathbb{T}$.

Dynamic equationson a time scale have enormous potential for applications such as in population dynamics. Forexample, it can model insect populations that are continuous while in season, die out in say winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population.
a nonoverlapping population.

In this study, we will generalize oscillation and nonoscillation criteria which are given for linear and nolinear differential equations to high order dynamic equations on time scales.

**Key Words:** Time Scales, Dynamic Equation, Oscillation

# 1. BÖLÜM

## GİRİŞ/AMAÇ VE KAPSAM

Bilim dünyası zaman skalası teorisi ile ilk kez Stefan Hilger'in 1988 deki doktora çalışması [1] ile tanıştı. Bu çalışmada Stefan Hilger sürekli ortam ve ayrık ortamdaki olayların analizini birleştirme üzerinde durdu. Bu çalışma uygulamalı bilimlerde etkisini göstermektedir. Zaman skalasındaki dinamik denklemler bunlardan birisidir. Zaman skalasındaki dinamik denklemler çalışması bu zorlukları ortadan kaldırmış ve matematikçileri hem diferensiyel hemde fark denklemleri için sonuçları ayrı ayrı ispat etme zahmetinden kurtarmıştır.

Zaman skalası ile ilgili temel kavramlar ve teoriler için Bohner ve Peterson [4-5] kitabıları referans olarak alınabilir.

Zaman skalasını yüzeysel olarak ifade edecek olursak reel sayıların boş olmayan keyfi kapalı alt kümesidir diyebiliriz. Zaman skalasının farklı seçimleri ile karşımıza farklı denklemler çıkar. Örneğin zaman skalası reel sayılar seçilirse adi diferensiyel denklemler, tam sayılar seçilirse fark denklemleri karşımıza çıkar. Son zamanlarda üzerinde birçok araştırmalar yapılan bu konu uygulama alanlarında büyük bir potansiyele sahiptir. Örneğin bir böcek türüne ait olan popülasyonu modelleyebiliriz. Bu türe ait yaşam süresi bir birim zaman olsun. Ayrıca bu tür ölmeden hemen önce yumurtlasın ve iki birim zaman sonra yavrular yumurtadan çıksın. Bu durumda örtüşmeyen bir popülasyon ortaya çıkar. Bu bize kapalı aralıkların birleşimini verir.

Bu araştırma projesinde ikinci ve üçüncü mertebeden nonlineer dynamic denklemlerin salınımlılık özellikleri incelenerek daha önce literatürde olmayan denklemler için salınımlılık kriterleri elde edilmiştir.

FBA-11-3391 nolu projeye ait sonuç raporunda sırasıyla 2. Bölümde, zaman skalasında temel kavramlar verilecek, 3. Bölümde elde edilen bulgular , 4. Bölümde tartışma ve sonuç verilecektir.

# 2. BÖLÜM

# GENEL BİLGİLER

## 2.1 ZAMAN SKALASI

Zaman skalasını basit bir şekilde ifade edecek olursak reel sayıların boş olmayan keyfi bir kapalı altkümesidir. Zaman skalasına reel sayılar, tam sayılar, doğal sayılar negatif olmayan tamsayılar, $[0,2]\cup[4,7]$ kapalı aralıkların birleşimi ve $[0,1]\cup\mathbf{N}$ kümesi örnek olarak verilebilir. Rasyonel sayılar, irrasyonel sayılar, kompleks sayılar ve $(0,1)$ açık aralığı ise zaman skalası olmayan kümelerdir.

Bu tez çalışmasında zaman skalası $\mathbb{T}$ sembolü ile gösterilecektir.

Zaman skalası, ilk kez ayrık ve sürekli ortamlardaki analizi birleştirmek için Stefan Hilger tarafından 1988'de ortaya atılmıştır. Gerçekten de ilerleyen bölümlerde $\mathbb{T}$ üzerinde tanımlı bir $f$ fonksiyonun $f^{\Delta}$ delta türevi reel sayılar ve tam sayılar için aşağıdaki gibi tanımlanacaktır:

**(i)** Eğer $\mathbf{T}=\mathbf{R}$ ise $f^{\Delta}=f'$ adi türev

**(ii)** Eğer $\mathbf{T}=\mathbf{Z}$ ise $f^{\Delta}=\Delta f$, ileri fark operatörü

olarak tanımlanır.

**Tanım 2.1.1** $\mathbf{T}$ bir zaman skalası olsun. Bir $t\in\mathbb{T}$ için

ileri sıçrama operatörü $\sigma:\mathbb{T}\to\mathbb{T}$

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$$

geri sıçrama operatörü $\rho : \mathbb{T} \to \mathbb{T}$

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$$

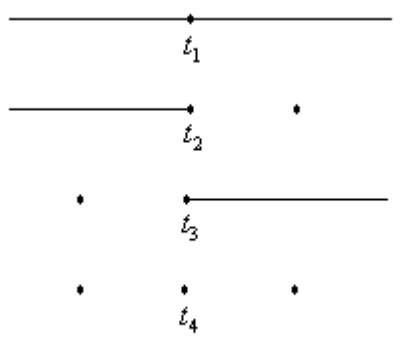
şeklinde tanımlanır.

$\sigma(t) > t$ ise $t$ noktasına sağda saçılımlı nokta, $\rho(t) < t$ ise $t$ noktasına solda saçılımlı nokta denir. Aynı anda solda ve sağda saçılımlı noktaya ise izole edilmiş nokta denir. $t < \sup \mathbb{T}$ ve $\sigma(t) = t$ ise $t$ noktasına sağda yoğun nokta, $t > \inf \mathbb{T}$ ve $\rho(t) = t$ ise $t$ noktasına solda yoğun nokta denir. Aynı anda solda yoğun ve sağda yoğun noktaya ise yoğun nokta denir. (Bakınız Tablo 1.1 ve Şekil 1.1)

**Tablo 1.1** Noktaların sınıflandırılması

| | |
|---|---|
| t sağdan saçılımlı | $\sigma(t) > t$ |
| t sağda yoğun | $\sigma(t) = t$ |
| t soldan saçılımlı | $\rho(t) < t$ |
| t solda yoğun | $\rho(t) = t$ |
| t izole edilmiş | $\rho(t) < t < \sigma(t)$ |
| t yoğun | $\rho(t) = t = \sigma(t)$ |

**Şekil 1.1** Noktaların şematik sınıflandırılması




$t_1$ sağda ve solda yoğun nokta

$t_2$ solda yoğun ve sağdan saçılımlı nokta

$t_3$ soldan saçılımlı ve sağda yoğun nokta

$t_4$ sağdan ve soldan saçılımlı nokta

($t_1$ yoğun nokta ve $t_4$ izole edilmiş nokta)

Ayrıca $\mu : \mathbb{T} \to [0, \infty)$ çekirdek fonksiyonu

$$\mu(t) := \sigma(t) - t$$

şeklinde tanımlanır.

Tanımdan herhangi bir $t \in \mathbb{T}$ için $\rho(t) \in \mathbf{T}$ ve $\sigma(t) \in \mathbf{T}$ olduğu açıktır. Şimdi $\mathbb{T}$ zaman skalasından türetilen $\mathbb{T}^{\kappa}$ kümesini tanımlayalım. Eğer $\mathbb{T}$ sola saçılımlı m maksimum değerine sahip ise o zaman $\mathbb{T}^{\kappa} = \mathbb{T} - \{m\}$. Diğer durumlarda ise $\mathbb{T}^{\kappa} = \mathbb{T}$. Özetlersek;

$$\mathbb{T}^{\kappa} = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & ; \quad \sup \mathbb{T} < \infty \\ \mathbb{T} & ; \quad \sup \mathbb{T} = \infty \end{cases}$$

şeklindedir. Ayrıca $f : \mathbb{T} \to ¡$ bir fonksiyon ise $f^{\sigma} : \mathbb{T} \to ¡$ fonksiyonu her $t \in \mathbb{T}$ için

$$f^{\sigma}(t) = (f o \sigma)(t)$$

şeklinde tanımlanır.

**Örnek 2.1.2** Farklı zaman skalaları için çekirdek fonksiyonunun nasıl değiştiğini görelim:

**(i)** $\mathbb{T} = \mathbb{R}$ alırsak, herhangi bir $t \in \mathbb{R}$ için

$$\sigma(t) = \inf\{s \in \mathbb{R} : s > t\} = \inf(t, \infty) = t\,,$$

$$\rho(t) := \sup\{s \in \mathbb{R} : s < t\} = \sup(-\infty, t) = t\,.$$

Böylece her $t \in \mathbb{R}$ noktası yoğundur. Dolayısıyla her $t \in \mathbb{R}$ için

$$\mu(t) = 0\,.$$

**(ii)** $\mathbb{T} = \mathbb{Z}$ alırsak, herhangi bir $t \in \mathbb{Z}$ için

$$\sigma(t) = \inf\{s \in \mathbb{Z} : s > t\} = \inf(t+1, t+2, t+3, ...) = t+1\,,$$
$$\rho(t) := \sup\{s \in \mathbb{Z} : s < t\} = \sup\{..., t-3, t-2, t-1\} = t-1\,.$$

Böylece $\rho(t) < t < \sigma(t)$ olduğundan her $t \in \mathbb{Z}$ noktası izole edilmiştir. Dolayısıyla her $t \in \mathbb{Z}$ için

$$\mu(t) := \sigma(t) - t = t + 1 - t = 1\,.$$

**(iii)** $\mathbf{T}=\{\sqrt{n}:n\in\mathbf{N}\}\cup\{0\}$ alırsak, herhangi bir $t\in\mathbb{T}$ için

$$\sigma(t)=\sigma(\sqrt{n})=\inf\{s\in\mathbf{T}:s>\sqrt{n}\}=\inf\{\sqrt{n+1},\sqrt{n+2},...\}=\sqrt{n+1}=\sqrt{t^2+1},$$

$$\rho(t)=\rho(\sqrt{n})=\sup\{s\in\mathbf{T}:s<\sqrt{n}\}=\sup\{...,\sqrt{n-2},\sqrt{n-1}\}=\sqrt{n-1}=\sqrt{t^2-1}$$

olur. Böylece $\rho(t)<t<\sigma(t)$ olduğundan her $t\in\mathbb{T}$ noktası izole edilmiştir. Dolayısıyla her $t\in\mathbb{T}$ için

$$\mu(t)=\sqrt{t^2+1}-t$$

elde edilir. Çekirdek fonksiyonu zaman skalasında çok önemli bir role sahiptir. Birçok formül, $\mu(t)$ çarpanını içeren terimlere sahiptir. Bu formüllerden bir tanesi de Riccati denklemidir. Genel zaman skalasında skaler Riccati denklemi [1]'de aşağıdaki formda verilmiştir.

$$z^{\Delta}+q(t)+\frac{z^2}{p(t)+\mu(t)z}=0$$

Bu denklemde $\mathbb{T}=\mathbb{R}$ alınırsa adi Riccati diferansiyel denklemini

$$z^{\Delta}+q(t)+\frac{1}{p(t)}z^2=0$$

elde ederiz. $\mathbb{T}=\mathbb{Z}$ alınırsa Riccati fark denklemini

$$z^{\Delta}+q(t)+\frac{z^2}{p(t)+z}=0$$

elde ederiz. $\mathbb{T}$ zaman skalasında [a,b] aralığını

$$[a,b]:=\{t\in\mathbb{T}:a\le t\le b\}$$

şeklinde tanımlayacağız. Diğer açık ve yarı açık aralıklarda benzer şekilde tanımlanır. Ayrıca $b$ solda yoğun nokta ise $[a,b]^{\kappa}=[a,b]$ ve $b$ soldan saçılımlı nokta ise $[a,b]^{\kappa}=[a,b)=[a,\rho(b)]$ elde edilir.

## 2.2 ZAMAN SKALASINDA TÜREV

**Tanım 2.2.1** $f:\mathbb{T}\to\mathbb{R}$ bir fonksiyon ve $t\in\mathbb{T}^{\kappa}$ olsun. Herhangi bir pozitif $\varepsilon$ sayısı için $t$'nin bir $N$ komşuluğunda her $s\in N$ için

$$\left|\left[f(\sigma(t))-f(s)\right]-f^{\Delta}(t)\left[\sigma(t)-s\right]\right|\le\varepsilon\left|\sigma(t)-s\right|$$

şartını sağlayan $f^{\Delta}(t)$ sayısına $f$ fonksiyonun $t$ noktasındaki delta türevi denir. Ayrıca her $t\in\mathbb{T}^{\kappa}$ için $f^{\Delta}(t)$ mevcutsa $f$ fonksiyonuna delta türevlenebilir denir. $f^{\Delta}:\mathbb{T}\to\mathbb{R}$ fonksiyonuna ise $f$ fonksiyonunun $\mathbb{T}^{\kappa}$ üzerindeki delta türevi denir.

**Örnek 2.2.2**

**(i)** $f:\mathbb{T}\to\mathbb{R}$ fonksiyonu her $t\in\mathbb{T}$ için $\alpha\in\square$ sabit sayı olmak üzere $f(t)=\alpha$ şeklinde tanımlansın. $f^{\Delta}(t)\equiv 0$ olduğunda herhangi bir pozitif $\varepsilon$ sayısı için,

$$\left|\left[f(\sigma(t))-f(s)\right]-0.\left[\sigma(t)-s\right]\right|=\left|\alpha-\alpha\right|=0\le\varepsilon\left|\sigma(t)-s\right|$$

şartı her $s\in\mathbb{T}$ için sağlanır.

**(ii)** $f:\mathbb{T}\to\mathbb{R}$ fonksiyonu her $t\in\mathbb{T}$ için $f(t)=t$ şeklinde tanımlansın. O zaman $f^{\Delta}(t)\equiv 1$ olur. Çünkü herhangi bir pozitif $\varepsilon$ sayısı için,

$$\left|\left[f(\sigma(t))-f(s)\right]-1.\left[\sigma(t)-s\right]\right|=\left|\sigma(t)-s-(\sigma(t)-s)\right|=0\le\varepsilon\left|\sigma(t)-s\right|$$

şartı her $s\in\mathbb{T}$ için sağlanır.

**Teorem 2.2.3** $f:\mathbb{T}\to\mathbb{R}$ bir fonksiyon ve $t\in\mathbb{T}^{\kappa}$ olsun. O zaman ;

**(i)** *f, t* noktasında türevlenebilir ise o zaman *f, t* noktasında süreklidir.

**(ii)** *f, t* noktasında sürekli ve *t* noktası sağdan saçılımlı ise o zaman *f, t* noktasında türevlenebilirdir ve bu durumda

$$f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}$$

olur.

**(iii)** $t$ noktası sağda yoğun ise o zaman $f$, $t$ noktasında türevlenebilirdir ancak ve ancak

$$\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

limiti mevcuttur. Bu durumda

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(\sigma(t))-f(t)}{\mu(t)}.$$

**iv)** $f$, $t$ noktasında türevlenebilir ise o zaman

$$f(\sigma(t))=f(t)+f^{\Delta}(t)\mu(t).$$

**Örnek 2.2.4** $f^{\Delta}(t)$ türevini bazı zaman skalası örnekleri için inceleyelim:

**(i)** $\mathbb{T}=\mathbb{R}$ alalım ve $f:\mathbb{R}\to\mathbb{R}$ fonksiyonu $t\in\mathbb{R}$ noktasında delta türevlenebilir olsun. $\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$ limiti mevcutsa $f$ fonksiyonunun $t$ noktasında türevlenebilir olduğunu ve bu limitin $f'(t)$ ile gösterildiğini türevin tanımından biliyoruz. O halde Teorem 1.2.3 (iii) şıkkından

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}=f'(t)$$

elde edilir. Böylece $\mathbb{T}=\mathbb{R}$ ise $f^{\Delta}(t)=f'(t)$ olur.

**(ii)** $\mathbb{T}=\mathbb{Z}$ ve $f:\mathbb{Z}\to\mathbb{R}$, $t\in\mathbb{Z}$ noktasında delta türevlenebilir olsun. O zaman

$$f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}=\frac{f(t+1)-f(t)}{1}=f(t+1)-f(t)=\Delta f(t)$$

elde edilir ki bu bilinen ileri fark operatöründen başkası değildir.

**(iii)** $\mathbf{T}=\{\sqrt{n}:n\in\mathbf{N}\}\cup\{0\}$ ve $f:\mathbb{Z}\to\mathbb{R}$, $t\in\mathbb{Z}$ noktasında delta türevlenebilir olsun. O zaman

$$f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}=\frac{f(\sqrt{t^2+1})-f(t)}{\sqrt{t^2+1}-t}$$

elde edilir.

**Teorem 2.2.5** $f,g:\mathbb{T}\to\mathbb{R}$ fonksiyonları $t\in\mathbb{T}^{\kappa}$ noktasında türevlenebilir olsun. O zaman

**(i)** $f+g:\mathbb{T}\to\mathbb{R}$, $t\in\mathbb{T}^{\kappa}$ noktasında türevlenebilirdir ve

$$\left(f+g\right)^{\Delta}\left(t\right)=f^{\Delta}(t)+g^{\Delta}(t).$$

**(ii)** Herhangi bir $\alpha$ sabiti için,

$$\left(\alpha f\right)^{\Delta}(t)=\alpha f^{\Delta}(t).$$

**(iii)** $fg:\mathbb{T}\to\mathbb{R}$, $t$ noktasında türevlenebilirdir ve

$$\left(fg\right)^{\Delta}(t)=f^{\Delta}(t)g(t)+f(\sigma(t))g^{\Delta}(t)=f(t)g^{\Delta}(t)+f^{\Delta}(t)g(\sigma(t)).$$

**(iv)** $f(t)f(\sigma(t))\neq 0$ ise $\frac{1}{f}$, $t$ noktasında türevlenebilirdir ve

$$\left(\frac{1}{f}\right)^{\Delta}(t)=-\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}.$$

**(v)** $g(t)g(\sigma(t))\neq 0$ ise $\frac{f}{g}$, $t$ noktasında türevlenebilirdir ve

$$\left(\frac{f}{g}\right)^{\Delta}(t)=\frac{f^{\Delta}(t)g(t)-f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

**Teorem 2.2.6** $\alpha$ bir sabit ve $m\in\mathbb{N}$ olsun.

**(i)** $f$ fonksiyonu $f(t)=(t-\alpha)^m$ şeklinde tanımlansın. O zaman

$$f^{\Delta}(t)=\sum_{v=0}^{m-1}\left(\sigma(t)-\alpha\right)^{v}\left(t-\alpha\right)^{m-1-v}.$$

**(ii)** $g$ fonksiyonu $g(t)=\dfrac{1}{(t-\alpha)^{m}}$ şeklinde tanımlansın. O zaman $\left(t-\alpha\right)\left(\sigma(t)-\alpha\right)\neq 0$ olmak üzere

$$g^{\Delta}(t)=-\sum_{v=0}^{m-1}\frac{1}{\left(\sigma(t)-\alpha\right)^{m-v}\left(t-\alpha\right)^{v+1}}.$$

**Örnek 2.2.7** $\mathbf{T}$ herhangi bir zaman skalası olsun. $f:\mathbf{T}\to\mathbf{R}$ fonksiyonu her $t\in\mathbf{T}$ için

**(i)** $f(t)=t^{2}$ ise

$$f^{\Delta}(t)=\sum_{v=0}^{1}\sigma(t)^{v}.t^{1-v}=t+\sigma(t)$$

$$=\begin{cases}2t & ;\ \mathbf{T}=\mathbf{R}\\ 2t+1 & ;\ \mathbf{T}=\mathbf{Z}\\ \sqrt{t^{2}+1}+t & ;\ \mathbf{T}=\{\sqrt{n}:n\in\mathbf{N}\}\cup\{0\}.\end{cases}$$

**(ii)** $f(t)=\left(t-1\right)^{3}$ ise

$$f^{\Delta}(t)=\sum_{v=0}^{2}(\sigma(t)-1)^{v}.\left(t-1\right)^{2-v}$$

$$=\left(t-1\right)^{2}+(\sigma(t)-1)\left(t-1\right)+(\sigma(t)-1)^{2}$$

$$=\begin{cases}3(t-1)^{2} & ;\ \mathbf{T}=\mathbf{R}\\ 3t^{2}-3t+1 & ;\ \mathbf{T}=\mathbf{Z}\\ 2t^{2}-3t+4+\sqrt{t^{2}+1}(t-3) & ;\ \mathbf{T}=\{\sqrt{n}:n\in\mathbf{N}\}\cup\{0\}.\end{cases}$$

**(iii)** $f(t)=\dfrac{1}{t}$ ise

$$f^{\Delta}(t) = -\frac{1}{t\sigma(t)}$$

$$= \begin{cases} -\dfrac{1}{t^2} & ; \quad \mathbf{T} = \mathbf{R} \\ -\dfrac{1}{t^2+t} & ; \quad \mathbf{T} = \mathbf{Z} \\ -\dfrac{1}{t\sqrt{t^2+1}} & ; \quad \mathbf{T} = \{\sqrt{n} : n \in \mathbf{N}\} \cup \{0\}. \end{cases}$$

**(iv)** $\frac{1}{t^2}$ ' nin türevini bulalım.

$$g^{\Delta}(t) = -\sum_{v=0}^{1} \frac{1}{\sigma(t)^{2-v}\left(t\right)^{v+1}} = -\frac{t+\sigma\left(t\right)}{\sigma(t)^2 t^2}$$

$$= \begin{cases} -\dfrac{2}{t^3} & ; \quad \mathbf{T} = \mathbf{R} \\ -\dfrac{2t+1}{t^2(t+1)^2} & ; \quad \mathbf{T} = \mathbf{Z} \\ -\dfrac{t+\sqrt{t^2+1}}{t^2(t^2+1)} & ; \quad \mathbf{T} = \{\sqrt{n} : n \in \mathbf{N}\} \cup \{0\}. \end{cases}$$

**Tanım 2.2.8** (Yüksek mertebeden türevler) $f : \mathbb{T} \to \mathbb{R}$ bir fonksiyon olsun. $f^{\Delta},\ \mathbb{T}^{\kappa^2} = \left(\mathbb{T}^{\kappa}\right)^{\kappa}$ üzerinde türevlenebilir olsun. O zaman $f^{\Delta\Delta} : \mathbb{T}^{\kappa^2} \to \mathbb{R}$'ya ikinci mertebeden delta türevi denir. Benzer şekilde n. mertebeden $f^{\Delta^n} : \mathbb{T}^{\kappa^n} \to \mathbb{R}$ delta türevleri de tanımlanabilir. Son olarak $t \in \mathbb{T}$ için $\sigma^2(t) = \sigma(\sigma(t))$ ve $\rho^2(t) = \rho(\rho(t))$ şeklinde tanımlansın. Benzer şekilde $n \in \mathbb{N}$ için $\sigma^n(t)$ ve $\rho^n(t)$ tanımlanabilir. Ayrıca

$$\rho^0(t) = \sigma^0(t) = t,\ f^{\Delta^0} = f \text{ ve } \mathbb{T}^{\kappa^0} = \mathbb{T}.$$

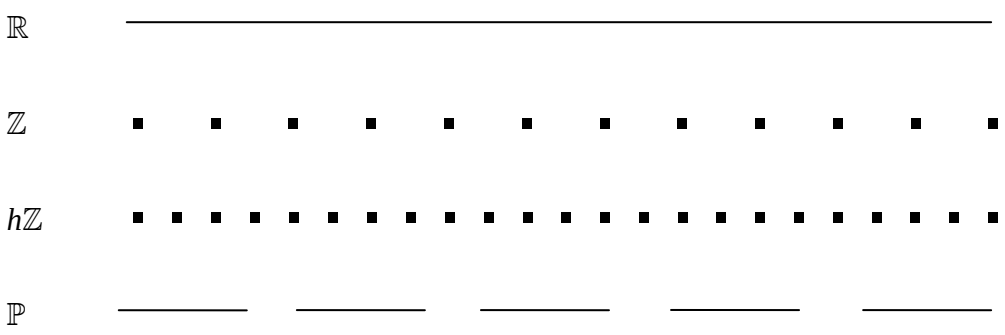
**Örnek 2.2.9** Genelde $f$ ve g iki kez türevlenebilir olsalar bile *fg* iki kez türevlenebilir olmayabilir. *f* ve g iki kez türevlenebilir ve $f^{\sigma}$ türevlenebilir olsun.

$$\left(fg\right)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta}$$

olduğu bilindiğine göre

$$\begin{aligned}(fg)^{\Delta\Delta} &= \left(f^{\Delta}g + f^{\sigma}g^{\Delta}\right)^{\Delta} \\ &= f^{\Delta\Delta}g + f^{\Delta^{\sigma}}g^{\Delta} + f^{\sigma^{\Delta}}g^{\Delta} + f^{\sigma^{\sigma}}g^{\Delta\Delta} \\ &= f^{\Delta\Delta}g + \left(f^{\Delta\sigma} + f^{\sigma\Delta}\right)g^{\Delta} + f^{\sigma\sigma}g^{\Delta\Delta}.\end{aligned}$$

**Şekil 1.3** Bazı Zaman Skalası Örnekleri




**Örnek 2.2.10** $q > 1$ olsun.

$$q^{\mathbb{Z}} := \left\{q^k : k \in \mathbb{Z}\right\} \text{ ve } \overline{q^{\mathbb{Z}}} := q^{\mathbb{Z}} \cup \{0\}$$

Burada zaman skalası olarak $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ alalım. O zaman $\forall t \in \mathbb{T}$ için

$$\sigma(t) = \inf\left\{q^n : n \in [m+1, \infty)\right\} = q^{m+1} = qq^m = qt,$$

$$\rho(t) = \sup\left\{q^n : n \in (-\infty, m-1]\right\} = q^{m-1} = q^{-1}q^m = \frac{t}{q},$$

$$\mu(t) = \sigma(t) - t = (q-1)t$$

olur. O halde 0 noktası sağda yoğun nokta, diğer noktalar ise izole edilmiş noktalardır. $f : \mathbb{T} \to \mathbb{R}$ fonksiyonu için

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(qt) - f(t)}{(q-1)t}, \ \forall t \in \mathbb{T} \setminus \{0\}$$

ve

$$f^{\Delta}(0)=\lim_{s\to 0}\frac{f(0)-f(s)}{0-s}=\lim_{s\to 0}\frac{f(s)-f(0)}{s}.$$

Şimdi $f$ fonksiyonunun $t\neq 0$ noktasındaki ikinci mertebeden türevine bakalım.

$$f^{\Delta\Delta}(t)=\frac{f^{\Delta}(\sigma(t))-f^{\Delta}(t)}{\mu(t)}$$

$$=\frac{f^{\Delta}(qt)-f^{\Delta}(t)}{(q-1)t}$$

$$=\frac{\dfrac{f(q^2t)-f(qt)}{q(q-1)t}-\dfrac{f(qt)-f(t)}{(q-1)t}}{(q-1)t}$$

$$=\frac{f(q^2t)-f(qt)-qf(qt)+qf(t)}{q(q-1)^2t^2}$$

$$=\frac{f(q^2t)-f(qt)(q+1)+qf(t)}{q(q-1)^2t^2}.$$

**Örnek 2.2.11** $\mathbb{T}=\mathbb{N}_0^2=\{n^2:\ n\in\mathbb{N}_0\}$ zaman skalasını düşünelim.

$$\sigma(n^2)=\inf\{s\in\mathbb{N}_0^2:s>n^2\}=\inf\{(n+1)^2:n\in\mathbb{N}_0\}=(n+1)^2,$$

$$\rho(n^2)=\sup\{s\in\mathbb{N}_0^2:s<n^2\}=\sup\{(n-1)^2:n\in\mathbb{N}_0\}=(n-1)^2,$$

$$\mu(n^2)=\sigma(n^2)-n^2=(n+1)^2-n^2=2n+1.$$

O halde $t\in\mathbb{T}$ için $\sigma(t)=\left(\sqrt{t}+1\right)^2$, $\rho(t)=\left(\sqrt{t}-1\right)^2$ ve $\mu(t)=1+2\sqrt{t}$ elde edilir. Böylece her $t\in\mathbb{T}$ noktası izole edilmiş noktadır. Ayrıca $f:\mathbb{T}\to\mathbb{R}$ fonksiyonu için

$$f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}=\frac{f(\left(\sqrt{t}+1\right)^2)-f(t)}{1+2\sqrt{t}},\ \forall t\in\mathbb{T}$$

elde edilir.

**Tablo 1.2** Zaman skalası örnekleri

| $\mathbb{T}$ | $\mu(t)$ | $\sigma(t)$ | $\rho(t)$ |
|---|---|---|---|
| $\mathbb{R}$ | $0$ | $t$ | $t$ |
| $\mathbb{Z}$ | $1$ | $t+1$ | $t-1$ |
| $h\mathbb{Z}$ | $h$ | $t+h$ | $t-h$ |
| $q^{\mathbb{N}}$ | $(q-1)t$ | $qt$ | $t/q$ |
| $\mathbb{N}_0^2$ | $2\sqrt{t}+1$ | $(\sqrt{t}+1)^2$ | $(\sqrt{t}-1)^2$ |

## 2.3. ZAMAN SKALASINDA İNTEGRAL

İntegrallenebilir fonksiyonların sınıflarını tanımlamadan önce aşağıdaki iki kavramın tanımını verelim.

**Tanım 2.3.1** $f:\mathbb{T}\to\mathbb{R}$ fonksiyonunun $\mathbb{T}$'deki bütün sağda yoğun noktalarda sağ limiti var ve bütün solda yoğun noktalarda sol limiti var ise $f$ fonksiyonuna düzenli (regulated) fonksiyon denir.

**Tanım 2.3.2** $f:\mathbb{T}\to\mathbb{R}$ fonksiyonunun $\mathbb{T}$'deki bütün sağda yoğun noktalarda sürekli ve bütün solda yoğun noktalarda sol limiti var ise $f$ fonksiyonuna **rd-sürekli** denir. Bu tezde $f:\mathbb{T}\to\mathbb{R}$ rd-sürekli fonksiyonlarının kümesi

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}\left(\mathbb{T},\mathbb{R}\right)$$

ile gösterilecektir. Türevlenebilir ve türevi rd-sürekli $f:\mathbb{T}\to\mathbb{R}$ fonksiyonların kümesi ise

$$C_{rd}^1 = C_{rd}^1(\mathbb{T}) = C_{rd}^1\left(\mathbb{T},\mathbb{R}\right)$$

şeklinde gösterilecektir.

**Teorem 2.3.3** $f:\mathbb{T}\to\mathbb{R}$ fonksiyonu için

**(i)** $f$ fonksiyonu sürekli ise o zaman $f$ fonksiyonu rd-süreklidir.

**(ii)** $f$ fonksiyonu rd-sürekli ise o zaman $f$ fonksiyonu da düzenlidir.

**(iii)** Sıçrama operatörü $\sigma$ rd-süreklidir.

**(iv)** $f$ fonksiyonu düzenli ve rd-sürekli ise o zaman $f^{\sigma}$ fonksiyonu da düzenli ve rd-süreklidir.

**(v)** $f$ fonksiyonu sürekli olsun. $g:\mathbb{T}\to\mathbb{R}$ fonksiyonu düzenli veya rd-sürekli ise $fog$ fonksiyonu da düzenli veya rd-süreklidir.

**Tanım 2.3.4** $f:\mathbb{T}\to\mathbb{R}$ sürekli fonksiyonu

**(i)** $D\subset\mathbb{T}^{\kappa}$ olmak üzere, $\mathbb{T}^{\kappa}\setminus D$ sayılabilirdir.

**(ii)** $D$ kümesi, $\mathbb{T}$'nin sağdan saçılımlı öğelerinin hiçbirini içermez.

**(iii)** $f$ fonksiyonu her $t\in D$ için türevlenebilir ise *f* fonksiyonuna $D$ türevlenebilme bölgesi ile birlikte önceden türevlenebilir denir.

**Örnek 2.3.5** $\mathbb{T}:=\mathbb{P}_{2,1}$ olsun. $f:\mathbb{T}\to\mathbb{R}$ fonksiyonu

$$f(t)=\begin{cases}0 & ;\quad t\in\bigcup_{k=0}^{\infty}[3k,3k+1]\\ t-3k-1 & ;\quad t\in[3k+1,3k+2],k\in\mathbb{N}_0\end{cases}$$

şeklinde tanımlansın. (Bakınız Şekil 1.2). O zaman $D$ türevlenebilme bölgesi

$$D:=\mathbb{T}\setminus\bigcup_{k=0}^{\infty}\{3k+1\}$$

ile birlikte *f* fonksiyonu önceden türevlenebilirdir.

**Teorem 2.3.6** $f$ düzenli olsun. O zaman $D$ türevlenebilme bölgesinde önceden türevlenebilir bir *F* fonksiyonu vardır öyle ki her $t\in D$ için

$$F^{\Delta}(t)=f(t).$$

**Tanım 2.3.7** Kabul edelim ki $f:\mathbb{T}\to\mathbb{R}$ fonksiyonu düzenli olsun. Teorem 1.3.9'daki gibi herhangi bir *F* fonksiyonuna $f$ fonksiyonunun ön antitürevi denir. Bir *f* düzenli fonksiyonunun belirsiz integralini aşağıdaki gibi tanımlarız:

$$\int f(t)\Delta t = F(t)+C$$

Burada $C$ bir sabit sayı ve *F* fonksiyonu *f* fonksiyonunun ön antitürevidir. Cauchy integralini de aşağıdaki gibi tanımlayalım: Her $r,s\in\mathbb{T}$ için

$$\int_r^s f(t)\Delta(t) = F(s)-F(r)\,.$$

Ayrıca her $t\in\mathbb{T}^\kappa$ için

$$\int f(t)\Delta t = F(t)+C$$

ise $F:\mathbb{T}\to\mathbb{R}$ fonksiyonuna $f:\mathbb{T}\to\mathbb{R}$ fonksiyonunun antitürevi denir.

**Örnek 2.3.8** Eğer $\mathbb{T}=\mathbb{Z}$ ise $a\neq 1$ bir sabit sayı olmak üzere,

$$\int a^t\Delta(t)$$

belirsiz integralini hesaplayalım.

$$\left(\frac{a^t}{a-1}\right)^\Delta = \Delta\left(\frac{a^t}{a-1}\right) = \frac{a^{t+1}-a^t}{a-1} = a^t$$

olduğundan $\int a^t\Delta(t) = \dfrac{a^t}{a-1}+C$ elde edilir.

**Teorem 2.3.9** Her rd-sürekli fonksiyon bir antitüreve sahiptir. Özel olarak $t_0\in\mathbb{T}$ ise o zaman $t\in\mathbb{T}$ için

$$F(t) := \int_{t_0}^{t} f(\tau)\Delta\tau$$

şeklinde tanımlanan $F$ fonksiyonu $f$ fonksiyonunun bir antitürevidir.

**Teorem 2.3.10** $f \in C_{rd}$ ve $t \in \mathbb{T}^{\kappa}$ ise o zaman

$$\int_{t}^{\sigma(t)} f(\tau)\Delta\tau = \mu(t) f(t)$$

**İ**

**Teorem 2.3.11** $f^{\Delta} \geq 0$ ise o zaman $f$ azalmayandır.

**Teorem 2.3.12** $a,b,c \in \mathbb{T},\ \alpha \in \mathbb{R}$ ve $f, g \in C_{rd}$ ise o zaman

**(i)** $\int_a^b [f(t)+g(t)]\Delta t = \int_a^b f(t)\Delta(t) + \int_a^b g(t)\Delta t;$

**(ii)** $\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t;$

**(iii)** $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t;$

**(iv)** $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t;$

**(v)** $\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t;$

**(vi)** $\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t;$

**(vii)** $\int_a^a f(t)\Delta t = 0;$

**(viii)** $[a,b)$ üzerinde $|f(t)| \leq g(t)$ ise o zaman

$$\left|\int_a^b f(t)\Delta t\right| \le \int_a^b g(t)\Delta t;$$

**(ix)** Her $a \le t \le b$ için, $f(t) \ge 0$ ise o zaman $\int_a^b f(t)\Delta t \ge 0$.

Burada (v) ve (vi) şıklarına **kısmi integrasyon** denir.

**Teorem 2.3.13** $a,b \in \mathbb{T}$ ve $f \in C_{rd}$ olsun.

**(i)** Eğer $\mathbb{T} = \mathbb{R}$ ise o zaman

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt .$$

**(ii)** $[a,b]$ aralığı sadece izole noktaları içeriyorsa o zaman

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t\in[a,b)} \mu(t) f(t) & ; \quad a < b \\ 0 & ; \quad a = b \\ -\sum_{t\in[a,b)} \mu(t) f(t) & ; \quad a > b \end{cases}$$

şeklindedir.

**(iii)** $h > 0$ olmak üzere, $\mathbb{T} = h\mathbb{Z} = \{hk : k \in Z\}$ ise o zaman

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{k=a/h}^{b/h} f(kh)h & ; \quad a < b \\ 0 & ; \quad a = b \\ -\sum_{k=b/h}^{a/h-1} f(kh)h & ; \quad a > b \end{cases}$$

şeklindedir.

**(iv)** $\mathbb{T} = \mathbb{Z}$ ise o zaman

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t=a}^{b-1} f(t) & ; \quad a < b \\ 0 & ; \quad a = b \\ -\sum_{t=b}^{a-1} f(t) & ; \quad a > b \end{cases}$$

şeklindedir.

**Tanım 2.3.14** $a \in \mathbb{T}$, $\sup \mathbb{T} = \infty$ ve $[a, \infty)$ aralığında *f,* rd-sürekli ise genelleştirilmiş integrali

$$\int_a^\infty f(t)\Delta t := \lim_{b \to \infty} \int_a^b f(t)\Delta t$$

şeklinde tanımlanır. Bu limit varsa genelleştirilmiş integral yakınsaktır. Eğer limit yoksa o zaman genelleştirilmiş integral ıraksaktır.

**Tanım 2.3.15.** Eğer aşikar olmayan x(t) çözümü sonsuz sayıda sıfırlara sahip ise bu çözüme salınımlı çözüm adı verilir. Denklemin tüm çözümleri salınımlı ise denkleme salınımlıdır denir.

# 3. BÖLÜM

# BULGULAR

Proje kapsamında literatürde yer almayan tipde yeni dinamik denklemleri ele alındı. İlk önce ikinci mertebeden lineer olmayan

$$(r(t)\Psi(x(t))x^{\Delta}(t))^{\Delta} + p(t)(x^{\Delta}(t))^{\sigma} + q(t)(fox^{\sigma}(t)) = 0$$

Dinamik denkleminin salınımlılık özellikleri incelendi . Burada $\Psi(x(t))$, $p(t)$; $q(t)$ ve $r(t)$ positif ve rd-sürekli fonksiyonlardır. Bu denkleme ilişkin aşağıdaki şartlar kabul edildi.

$(H1)$ $f : R \to \mathrm{R}$ , $f(u)=u \geq K > 0$, $uf(u) > 0$ , $u \neq 0$ ve $K > 0$,

$(H2)$ $0 < c_1 \leq \Psi(v) \leq c_2$ , $v$ herhangi bir fonksiyon,

$(H3)$ $\int_{t_0}^{\infty}\left(\frac{1}{cr(s)} e_{\frac{-p(s)}{cr(s)}}(s,t_0)\right)\Delta s = \infty$ ,bazı c>0 için.

Elde edilen sonuçlar uluslararası bir dergide yayınlandı.

Daha sonra yine ikinci mertebeden

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + p(t)\left(x^{\Delta}(t)\right)^{\gamma} + f(t,x(g(t))) = 0,$$

Dinamik denklemi ele alındı ve keyfi zaman skalasında Riccati dönüşüm tekniği ile çözümlerinin davranışı incelendi. Elde edilen sonuçlar SCI(A) sınıfı bir dergide yayınlandı. Üçüncü ve dördüncü olarak

$$(r(t)\Psi(x^{\Delta}(t))^{\Delta} + p(t)\Psi(x^{\Delta}(t)) + q(t)x^{\sigma}(t) = 0$$

$$(r(t)\Psi(x^{\Delta}(t))^{\Delta} + p(t)\Psi(x^{\Delta}(t)) + q(t)f(x^{\sigma}(t)) = 0, t \in \mathbb{T},$$

dinamik denklemleri ele alındı. Riccati dönüşüm ve İntegral averages tekniği ile denkleme ilişkin salınımlılık şartları elde edildi. Elde edilen sonuçlar uluslararası bir dergilerde yayınlandı.

Elde edilen sonuçların bazıları Uluslararası ve ulusal sempozyumlarda bildiri olarak sunuldu. Bu makale ve bildiriler aşağıda sıralanmıştır.

### 3.1 Makaleler

1- Şenel, M. T ,Oscillation Theorems for Dynamic Equation on Time Scales, Bulletin Of Mathematical Analysis and Applications, 3,4 (2011)101-105.

2- Şenel, M. T, Kamenev-Type Oscillation Criteria for the Second-Order Nonlinear Dynamic Equations with Damping on Time Scales, Abstract and Applied Analysis Volume 2012, Article ID 253107, 18 pages doi:10.1155/2012/253107. (SCI)

3- Şenel, M. T., "Oscillation results of second order damped non-linear dynamic equation on time scales", Appl. Math. Inf. Sci. Lett. 1, No. 1, 29-34 (2013) (SCI)

4- Şenel, M. T., Oscillation theorems for second-order damped dynamic equation on timescales, AIP Conference Proceedings , 251-254, 2012 , (SCI)
DOI: 10.1063/1.4747688.

5- Şenel, M. T., Behavior of solutions of a third-order dynamic equation on time scales, Journal of Inequalities and Applications, İncelemede.

### 3.2. Bildiriler

1- Şenel,M. T., "Oscillation Theorems for Second-Order Nonlinear Dynamic Equation on Time Scales", International Conference On Applied Analysis And Algebra ,Yıldız Technical University 145-146 pp., İstanbul, Türkiye, Haziran 2011

2- Şenel, M. T., “Zaman Skalasında 3. Mertebeden Nötral Dinamik Denklemlerin Çözümlerinin Davranışı” ,XXV. Ulusal Matematik Sempozyumu, 82-83 pp., Niğde, Türkiye, Eylül 2012.

3- Şenel, M. T., “Oscillation Theorems for Second-Order Damped Dynamic Equation on Time Scales”, First International Conference on Analysis and Applied Mathematics (ICAAM-2012) GUMUSHANE-September 19,2012.

# 4 BÖLÜM

# TARTIŞMA VE SONUÇ

Bu proje çalışmasında başarılı sonuçlar elde edilmiş olup daha önce literatürde yer almayan denklemlere ilişkin orijinal sonuçlar elde edilmiştir. Zaman skalası kavramı 1990 yıllarda tanımlanmış yeni bir kavram olup zaman skalasında dinamik denklemlerin incelenmesi literatüre önemli bir katkı sağlamaktadır. Bu projede elde edilen sonuçlar bundan sonra yapılacak olan çalışmalara ışık tutacaktır. Çünkü daha bir çok adi diferensiyel denklem için elde edilen sonuçlar zaman skalasında genelleştirebilir.

## 5. KAYNAKLAR

[1] S. Hilger, Analysis on measure chains a unified approach to continuous and discrete calculus, Results Math. 18 , 18-56, (1990).

[2] R. P. Agarwal, D. O'Regan and S. H. Saker, Philos- type oscillation criteria for second order half linear dynamic equations, Rocky Mountain J. Math. 37 (2007) 1085–1104.

[3] R. P. Agarwal, S. L. Shien and C. C. Yeh, Oscillation criteria for second-order retarded differential equations, Math. Comp. Modelling 26 (1997), 1–11.

[4] M. Bohner and A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser, Boston, 2001.

[5] M. Bohner, A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston,2003.

[6] L. Erbe, T. S. Hassan, A. Peterson and S. H. Saker, Oscillation criteria for half-linear delay dynamic equations on time scales, Nonlinear Dynam. Sys. Th.(Submitted).

[7] L. Erbe, T. S. Hassan, A. Peterson and S. H. Saker, Oscillation criteria for sublinear half-linear delay dynamic equations on time scales, J. Comp. Appl. Math. (Submitted).

[8] L. Erbe, A. Peterson and S. H. Saker, Hille-Kneser-type criteria for second-order dynamic equations on time scales, Adv. Diff. Eq. 2006 (2006) 1-18.

[9] L. Erbe, A. Peterson and S. H. Saker, Oscillation criteria for second-order nonlinear dynamic equations on time scales, J. London Math. Soc. 76 (2003) 701–714.

[10] L. Erbe, A. Peterson and S. H. Saker, Oscillation criteria for second-order nonlinear delay dynamic equations on time scales, J. Math. Anal. Appl. 333 (2007) 505–522.

[11] W.B. Fite, Concerning the zeros of solutions of certain differential equations, Trans. Amer. Math. Soc. 19 (1917) 341–352.

[12] S. Hilger, Analysis on measure chains — a unified approach to continuous and discrete calculus, Results Math. 18 (1990) 18–56.

[13] V. Kac and P. Cheung, Quantum Calculus, Universitext, 2002.

[14] J. V. Manojlovic, Oscillation criteria for second-order half-linear differential equations, Math. Comp. Mod. 30 (1999) 109-119.

[15] Y. Sahiner, Oscillation of second-order delay dynamic equations on time scales, Nonlinear Analysis: Th. Meth. Appl. 63 (2005) 1073–1080.

[16] S. H. Saker, Oscillation criteria of second-order half-linear dynamic equations on time scales, J. Comp. Appl. Math. 177 (2005) 375–387.

[17] E. Thandapani, K. Ravi, and J. Graef, Oscillation and comparison theorems for half-linear second order difference equations, Comp. Math. Appl. 42 (2001), 953–960.

[18] A. Wintner, On the nonexistence of conjugate points, Amer. J. Math. 73 (1951) 368–380.

[19] B. G. Zhang and S. Zhu, Oscillation of second-order nonlinear delay dynamic equations on time scales, Comp. Math. Appl. 49 (2005) 599–609.

# 6. EKLER

## Proje Çalışmasından Elde Edilen Veriler Kullanılarak Üretilmiş Yayınlar

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# Oscillation results of second order damped non-linear dynamic equation on time scales

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**Abstract:** This paper concerns the oscillation of solutions to second order non-linear dynamic equation with damping

$$(r(t)\Psi(x^{\Delta}(t))^{\Delta}+p(t)\Psi(x^{\Delta}(t))+q(t)x^{\sigma}(t)=0$$

on a time scale $\mathbb{T}$ which is unbounded above. $r(t)$, $p(t)$ and $q(t)$ are positive rd-continuous functions. $\Psi:\mathbb{T}\to\mathbb{R}$ is rd-continuous functions. Our results are new and different many known results for second order dynamic equations.

**Keywords:** Oscillation, Dynamic equations, Time scales

## 1. Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his PhD thesis in 1988 in order to unify continuous and discrete analysis (see [1]). Since Stefan Hilger formed the definition of derivatives and integrals on time scales, several authors have expounded on various aspects of the new theory, see the paper by Agarwal, et al. ([2]) and the references cited . A book on the subject of time scales by Bohner and Peterson [3] summarizes and organizes much of time scale calculus.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since we are interested in the oscillatory of solutions near infinity, we assume that $sup\mathbb{T}=\infty$, and define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$. We assume that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{T}$.

In this paper we shall study the oscillations of the following non-linear second order dynamic equations with damping

$$(r(t)\Psi(x^{\Delta}(t))^{\Delta}+p(t)\Psi(x^{\Delta}(t))+q(t)x^{\sigma}(t)=0, \qquad (1)$$

where $p(t),q(t)$ and $r(t)$ are positive rd-continuous functions.

In the last few years, much interest has focused on obtaining sufficient conditions for the oscillation/nonoscillation of solutions of different classes of dynamic equations on time scales, and we refer the reader to the papers [4-21].

Agarwal et al. ([4]), have considered the second order perturbed dynamic equation

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta}+F(t,x(t))=G(t,x(t),x^{\Delta}(t)), \qquad (2)$$

where $\gamma\in\mathbb{N}$ is odd and they have interested in asymptotic behavior of solutions of equation (2). In [5], Saker and et al. considered the non-linear dynamic equation

$$(a(t)x^{\Delta}(t))^{\Delta}+p(t)x^{\Delta^{\sigma}}(t)+q(t)f(x^{\sigma}(t))=0$$

when $a(t),p(t),r(t)$ are positive rd-continuous functions. They gave some sufficient conditions for oscillation.
The authors supposed that $uf(u)>0, f(u)/u\geq K>0$ and $f'(u)\geq k$ for $u\neq 0$ .

In this paper, by employing the Riccati transformation technique we will establish some sufficient conditions for the oscillation of (1). The paper is organized as follows: In Section 2, we develop the Riccati transformation technique to give some sufficient conditions for the oscillation of all solutions of (1). In Section 3, we establish some sufficient conditions for oscillation of Eq. (1) with $p(t)=0$.

We will use some of following assumptions:
$(H_1)$ $r(t),p(t)$, and $q(t)$ are positive real-valued rd-functions,
$(H_2)\Psi:\mathbb{T}\to\mathbb{R}$, $\frac{\Psi(u)}{|u|}\geq\kappa$ for $\kappa>0$, $u\neq 0$,
$(H_3)$ $\int_{t_0}^{\infty}(\frac{1}{r(t)}e_{-\frac{p}{r}}(t,t_0))\Delta t=\infty$ .

Our attention is restricted to those solutions of (1) which

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exist on some half-line $[t_x,\infty)$ and satisfy $sup\{|x(t)| : t > \mathbb{T}\} > 0$ for any $T \geq t_x$ . We assume the standing hypothesis that (1) does possess such solutions. A solution *x(t)* of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

## 2. Main results

**Theorem 2.1.** Assume that $(H_1)-(H_3)$ holds. Furthermore, assume that there exist a positive real rd-functions differentiable functions $z(t)$ such that

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[z(s)q(s)-\frac{\kappa r(s)A^2(s)}{4z(s)}\right]\Delta s=\infty, \tag{3}$$

where

$$A(t)=\left[z^{\Delta}(t)-\frac{z(t)p(t)}{r(t)}\right],$$

then every solution of (1) is oscillatory.

**Proof.** Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (1). Without loss of generality, we may assume that $x(t) > 0$ for $t \geq t_1 > t_0$. We shall consider only this case, since in view of $(H_2)$, the proof of the case when $x(t)$ is eventually negative is similar. Now, we claim that $x^{\Delta}(t)$ has a fixed sign on the interval $[t_2,\infty)$ for some $t_2 \geq t_1$. From (1), since $q(t) > 0$ , we have

$$(r(t)\Psi(x^{\Delta}(t))^{\Delta}+p(t)\Psi(x^{\Delta}(t))=-q(t)x^{\sigma}(t)<0,$$

i.e.,

$$(r(t)\Psi(x^{\Delta}(t))^{\Delta}+p(t)\Psi(x^{\Delta}(t))<0.$$

By setting

$$y(t)=r(t)\Psi(x^{\Delta}(t)),$$

we immediately see that,

$$y^{\Delta}(t)+\frac{p(t)y(t)}{r(t)}<0,$$

which implies that

$$\left(y(t)e_{-\frac{p}{r}}\right)^{\Delta}<0.$$

Then $y(t)e_{-\frac{p}{r}}$ is decreasing and thus $y(t)$ is eventually of one sing. Then $x^{\Delta}(t)$ has a fixed sing for all sufficiently large t and we have one of the following:
First, we consider $x^{\Delta}(t) \geq 0$ on $[t_2,\infty)$ for some $t_2 \geq t_1$. Then in view of (1) we have

$$x(t)>0, x^{\Delta}(t)\geq 0, (r(t)\Psi(x^{\Delta}(t))^{\Delta}\leq 0, t\geq t_2. \tag{4}$$

Define the function $w(t)$ by Riccati substitution

$$w(t):=z(t)\frac{r(t)\Psi(x^{\Delta}(t))}{x(t)}, t\geq t_2 \tag{5}$$

Then $w(t) > 0$, and satisfies

$$w^{\Delta}(t)=\left[r(t)\Psi(x^{\Delta}(t))\right]^{\sigma}\left[\frac{z(t)}{x(t)}\right]^{\Delta}+\frac{z(t)}{x(t)}\left[r(t)\Psi(x^{\Delta}(t))\right]^{\Delta}$$

In view of (1) and (5), we see that for $t \geq t_3$

$$\begin{aligned} w^{\Delta}(t)=&\frac{z^{\Delta}(t)-z(t)x^{\Delta}(t)}{x(t)x^{\sigma}(t)}\left[r(t)\Psi(x^{\Delta}(t))\right]^{\sigma}\\ &+\frac{z(t)}{x(t)}\left[-p(t)\Psi(x^{\Delta}(t))-q(t)x^{\sigma}(t)\right] \end{aligned} \tag{6}$$

However from (4),

$$r(t)\Psi(x^{\Delta}(t))\geq(r(t)\Psi(x^{\Delta}(t)))^{\sigma},\ x^{\sigma}(t)\geq x(t). \tag{7}$$

Using (7) and $(H_2)$ in (6), we have

$$\begin{aligned} w^{\Delta}(t)\leq z^{\Delta}(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)}&-\frac{z(t)}{x^{\sigma}(t)}p(t)\Psi(x^{\Delta}(t))-z(t)\frac{q(t)x^{\sigma}(t)}{x^{\sigma}(t)}\\ &-z(t)\frac{x^{\Delta}(t)}{(x^{\sigma}(t))^2}[r(t)\Psi(x^{\Delta}(t))]^{\sigma} \end{aligned}$$

$$\begin{aligned} w^{\Delta}(t)\leq z^{\Delta}(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)}-\frac{z(t)p(t)}{r(t)}\frac{w^{\sigma}(t)}{z^{\sigma}(t)}&-z(t)q(t)\\ &-z(t)\frac{(w^{\sigma}(t))^2}{\kappa(z^{\sigma}(t))^2r(t)} \end{aligned}$$

$$\begin{aligned} w^{\Delta}(t)\leq -z(t)q(t)+&\left[z^{\Delta}(t)-\frac{z(t)p(t)}{r(t)}\right]\frac{w^{\sigma}(t)}{z^{\sigma}(t)}\\ &-z(t)\frac{(w^{\sigma}(t))^2}{\kappa(z^{\sigma}(t))^2r(t)}, \end{aligned} \tag{8}$$

$$w^{\Delta}(t)\leq -z(t)q(t)+A(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)}-z(t)\frac{(w^{\sigma}(t))^2}{\kappa(z^{\sigma}(t))^2r(t)}, \tag{9}$$

where

$$A(t)=\left[z^{\Delta}(t)-\frac{z(t)p(t)}{r(t)}\right].$$

Then

$$\begin{aligned} w^{\Delta}(t)\leq&-z(t)q(t)+\frac{\kappa r(t)A^2(t)}{4z(t)}\\ &-\left[\sqrt{\frac{z(t)}{\kappa r(t)}}\frac{w^{\sigma}(t)}{z^{\sigma}(t)}-\frac{1}{2}\sqrt{\frac{\kappa r(t)}{z(t)}}A(t)\right]^2, \end{aligned}$$

$$w^{\Delta}(t)\leq z(t)q(t)-\frac{\kappa r(t)A^2(t)}{4z(t)}.$$

Integration from $t_3$ to t, we obtain

$$w(t)-w(t_3)\leq-\int_{t_3}^{t}\left[z(s)q(s)-\frac{\kappa r(s)A^2(s)}{4z(s)}\right]\Delta s$$

which yields

$$\int_{t_3}^{t}\left[z(s)q(s)-\frac{\kappa r(s)A^2(s)}{4z(s)}\right]\Delta s\leq w(t_3)-w(t)<w(t_3), t\geq t_3$$

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for all large t. This is contrary to (3).
Next, we consider $x^{\Delta}(t) < 0$ for $t \geq t_2 \geq t_1$ .
Define the function $u(t) = -r(t)\Psi(x^{\Delta}(t))$. The from (1) and $(H_3)$, we have

$$u^{\Delta}(t) + \frac{p(t)}{r(t)}u(t) \geq 0 \Rightarrow u(t) \geq u(t_2)e_{-\frac{p}{r}}(t,t_2),$$

Thus

$$-r(t)\Psi(x^{\Delta}(t)) \geq u(t_2)e_{-\frac{p}{r}}(t,t_2).$$

$$\Psi(x^{\Delta}(t)) \leq -u(t_2)\left(\frac{1}{r(t)}e_{-\frac{p}{r}}(t,t_2)\right).$$

from $(H_3)$ there is a $\kappa > 0$, so that

$$\kappa x^{\Delta}(t) \leq -u(t_2)\left(\frac{1}{r(t)}e_{-\frac{p}{r}}(t,t_2)\right). \tag{10}$$

Integrating (10) from $t_2$ to t, we have

$$x(t) - x(t_2) \leq \frac{r(t_2)\Psi(x(t_2))}{\kappa}\int_{t_2}^{t}\left(\frac{1}{r(t)}e_{-\frac{p}{r}}(t,t_2)\right)\Delta s.$$

$$x(t) \leq x(t_2) + \frac{r(t_2)\Psi(x(t_2))}{\kappa}\int_{t_2}^{t}\left(\frac{1}{r(t)}e_{-\frac{p}{r}}(t,t_2)\right)\Delta s.$$

so condition $(H_3)$ implies that x(t) is eventually negative, which is a contradiction. The proof is complete.

**Corollary 2.2.** Assume that $(H_1)-(H_3)$ hold. If

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[q(s) - \frac{\kappa p^2(s)}{4r(s)}\right]\Delta s = \infty \tag{11}$$

then every solution (1) is oscillatory.
**Example 2.3.** Consider the dynamic equation

$$\left(t\Psi(x^{\Delta}(t))\right)^{\Delta} + \left(\Psi(x^{\Delta}(t))\right) + \frac{1}{t}x^{\sigma}(t) = 0, \;\; t > 0$$

where $r(t) = t,\; p(t) = 1,\; q(t) = \frac{1}{t}$,
$\Psi(x^{\Delta}(t)) = (x^{\Delta}(t))^{2k+1}$, $k \in \mathbb{N}$. All conditions of Corollary 2.2 and $(H_1)-(H_3)$ are satisfied. Hence it is oscillatory.

**Corollary 2.4.** Assume that $(H_1)-(H_3)$ hold. If

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[s^{\gamma}q(s) - \frac{\kappa(r(s)(s^{\gamma})^{\Delta} - s^{\gamma}p(s))^2}{4r(s)}\kappa s^{-\gamma}\right]\Delta s = \infty \tag{12}$$

then every solution (1) is oscillatory.

**Corollary 2.5.** Assume that $(H_1)-(H_3)$ hold. If

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[Z(s,t_0)q(s) - \frac{\kappa r(s)}{4Z(s,t_0)}\left((Z(s,t_0))^{\Delta} - \frac{Z(s,t_0)p(s)}{r(s)}\right)^2\right]\Delta s = \infty,$$

where $Z(t,t_0) = \int_{t_0}^{t}\frac{1}{r(s)}\Delta s$, then every solution (1) is oscillatory.
Now, let us introduce the class of functions $\mathbb{R}$ which will be extensively used in the sequel. Let $\mathbb{D}_0 \equiv \{(t,s) \in \mathbb{T}^2 : t > s \geq t_0\}$ and $\mathbb{D} \equiv \{(t,s) \in \mathbb{T}^2 : t \geq s \geq t_0\}$. The function $H \in C_{rd}(\mathbb{D},\mathbb{R})$ is said belongs to the class $\mathfrak{R}$ if
(i) $H(t,t) = 0,\; t \geq t_0,\; H(t,s) > 0$, on $\mathbb{D}_0$,
(ii) H has a continuous $\Delta$-partial derivative $H_s^{\Delta}(t,s)$ on $\mathbb{D}_0$ with respect to the second variable.(H is rd-continuous function if H is rd-continuous function in t and s.)

**Theorem 2.6.** Assume that $(H_1)-(H_3)$ hold.Let z(t) be positive real rd-functions differentiable function and let $H : \mathbb{D} \to \mathbb{R}$ be rd-continuous function such that H belongs to the class $\mathfrak{R}$ and where

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)z(s)q(s) - \frac{\kappa r(s)(\varphi(t,s))^2}{4z(s)H(t,s)}\right]\Delta s = \infty, \tag{13}$$

$$\varphi(t,s) = z^{\sigma}(s)H_s^{\Delta}(t,s) + H(t,s)A(s).$$

Then every solution of (1) is oscillatory.

**Proof.** Suppose to the contrary that x(t) is a nonoscillatory solution of (1) and let $t_1 \geq t_0$ be such that $x(t) \neq 0$ for all $t \geq t_1$ , so without loss of generality, we may assume that x(t) is an eventually positive solution of (1) with $x(t) > 0$ for all $t \geq t_1$ sufficiently large. In view of Theorem 2.1 we see that $x^{\Delta}(t)$ is eventually negative or eventually positive. If $x^{\Delta}(t)$ is eventually negative, we are then back to second case of Theorem 2.1 and we obtain a contradiction. If $x^{\Delta}(t)$ is eventually positive, we assume that there exists $t_2 \geq t_1$ such that $x^{\Delta}(t) \geq 0$ for $t_2 \geq t_1$ and proceed as in the proof of first of Theorem 2 . From (9), it follows that

$$w^{\Delta}(t) \leq -z(t)q(t) + A(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - z(t)\frac{(w^{\sigma}(t))^2}{\kappa(z^{\sigma}(t))^2 r(t)}, \tag{14}$$

we multiply to (14) to $H(t,s)$ then

$$H(t,s)w^{\Delta}(t) \leq -H(t,s)z(t)q(t) + H(t,s)A(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - H(t,s)z(t)\frac{(w^{\sigma}(t))^2}{\kappa(z^{\sigma}(t))^2 r(t)},$$

$$H(t,s)z(t)q(t) \leq -H(t,s)w^{\Delta}(t) + H(t,s)A(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - H(t,s)z(t)\frac{(w^{\sigma}(t))^2}{\kappa(z^{\sigma}(t))^2 r(t)},$$

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Using the integration by parts formula, we have

$$\int_{t_2}^{t} H(t,s)z(s)q(s)\Delta s \le -H(t,t)w(t)+H(t,t_2)w(t_2) + \int_{t_2}^{t} H_s^{\Delta}(t,s)w^{\sigma}(s)\Delta s + \int_{t_2}^{t} H(t,s)A(s)\frac{w^{\sigma}(s)}{z^{\sigma}(s)}\Delta s - \int_{t_2}^{t} H(t,s)z(s)\frac{((w^{\sigma}(s))^2}{\kappa(z^{\sigma}(s))^2 r(s)}\Delta s,$$

where $H(t,t)=0$, we obtain

$$\int_{t_2}^{t} H(t,s)z(s)q(s)\Delta s \le H(t,t_2)w(t_2)+\int_{t_2}^{t}\left[z^{\sigma}(s)H_s^{\Delta}(t,s) + H(t,s)A(s)\right]\frac{w^{\sigma}(s)}{z^{\sigma}(s)}\Delta s - \int_{t_2}^{t} H(t,s)z(s)\frac{((w^{\sigma}(s))^2}{\kappa(z^{\sigma}(s))^2 r(s)}\Delta s,$$

$$\int_{t_2}^{t} H(t,s)z(s)q(s)\Delta s \le H(t,t_2)w(t_2)+\int_{t_2}^{t}\varphi(t,s)\frac{w^{\sigma}(s)}{z^{\sigma}(s)}\Delta s - \int_{t_2}^{t} H(t,s)z(s)\frac{((w^{\sigma}(s))^2}{\kappa(z^{\sigma}(s))^2 r(s)}\Delta s.$$

Therefore, by completing the square as in Theorem 2.1, we obtain

$$\int_{t_2}^{t} H(t,s)z(s)q(s)\Delta s \le H(t,t_2)w(t_2) + \int_{t_2}^{t}\frac{\kappa r(s)}{4z(s)H(t,s)}\varphi^2(t,s)\Delta s - \int_{t_2}^{t}\left[\sqrt{\frac{H(t,s)z(s)}{\kappa r(s)}}\frac{w^{\sigma}(s)}{z^{\sigma}(s)} - \frac{1}{2}\sqrt{\frac{\kappa r(s)}{z(s)H(t,s)}}\,\varphi(t,s)\right]^2\Delta s.$$

Hence, we obtain

$$\int_{t_2}^{t} H(t,s)z(s)q(s)\Delta s \le H(t,t_2)w(t_2) + \int_{t_2}^{t}\frac{\kappa r(s)}{4z(s)H(t,s)}\varphi^2(t,s)\Delta s.$$

Then for all $t \ge t_2$, we have

$$\int_{t_2}^{t}\left[H(t,s)z(s)q(s) - \frac{\kappa r(s)}{4z(s)H(t,s)}\varphi^2(t,s)\right]\Delta \le H(t,t_2)w(t_2)$$

and this implies that

$$\limsup_{t\to\infty}\frac{1}{H(t,t_2)}\int_{t_2}^{t}\left[H(t,s)z(s)q(s) - \frac{\kappa r(s)}{4z(s)H(t,s)}\varphi^2(t,s)\right]\Delta s \le w(t_2),$$

which contradicts (13). The proof is complete.

The consequences of Theorem 2.6, we get the following.

**Corollary 2.7.** Suppose that the assumptions of Theorem 2.6 hold. If

$$\limsup_{t\to\infty}\frac{1}{H(t,t_2)}\int_{t_2}^{t} H(t,s)\left[q(s) - \frac{\kappa r(s)}{4z(s)}\left(\frac{H_s^{\Delta}(t,s)}{H(t,s)} - \frac{p(s)}{r(s)}\right)^2\right]\Delta s = \infty,$$

then every solution of (1) is oscillatory.

**Corollary 2.8.** Let the assumption (13) in Theorem 2.6 be replaced by

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)z(s)q(s) = \infty,$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[\frac{\kappa r(s)}{4z(s)H(t,s)}\left(H(t,s)A(s) + z^{\sigma}(s)H_s^{\Delta}(t,s)\right)^2\right]\Delta s < \infty,$$

then every solution of (1) is oscillatory.

**Remarks 2.9.** [3, Remarks 2.3] Let $H(t,s) = (t-s)^n$ , $(t,s) \in \mathbb{D}$ with $n > 1$, we see that $H$ belongs to the class $\mathfrak{R}$. Hence

$((t-s)^n)^{\Delta} \le -n(t-\sigma(s))^{n-1}$.

**Corollary 2.10.** Assume that $(H_1)-(H_3)$ hold.Let $z(t)$ be positive real rd-functions differentiable function . If

$$\limsup_{t\to\infty}\frac{1}{t^n}\int_{t_0}^{t}\left[(t-s)^n z(s)q(s) - \frac{\kappa r(s)\phi^2(t,s)}{4z(s)(t-s)^n}\right]\Delta s = \infty,$$

where

$\phi(t,s) = (t-s)^n A(s) + nz^{\sigma}(t)(t-\sigma(s))^{n-1}, t \ge s \ge t_0,\ n > 1,$

then equation (1) is oscillatory on $[t_0,\infty)$.

## 3. Equation (1) with $p(t) = 0$.

We establish some sufficient conditions for oscillation of Eq. (1) with $p(t) = 0$.

**Theorem 3.1** Assume that $(H_1)-(H_3)$ hold. Furthermore, assume that there exists a positive real rd-continuous function $z(t)$ such that

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[z(s)q(s) - \frac{\kappa r(s)}{4z(s)}A^2(s)\right]\Delta s = \infty \qquad (15)$$

then every solution of Eq. (1) is oscillatory.

**Proof.** Suppose to the contrary that $x(t)$ is a nonoscilla-

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tory solution of (1) and let $t_1 \geq t_0$ be such that $x(t) \neq 0$ for all $t \geq t_1$, so without loss of generality, we may assume that $x(t)$ is an eventually positive solution of (1) with $x(t) > 0$ for all $t \geq t_1$ sufficiently large. In view of Theorem 2.1 we see that $x^{\Delta}(t)$ is eventually negative or eventually positive. If $x^{\Delta}(t)$ is eventually negative, we are then back to second case of Theorem 2.1 and we obtain a contradiction. If $x^{\Delta}(t)$ is eventually positive, we assume that there exists $t_2 \geq t_1$ such that $x^{\Delta}(t) \geq 0$ for $t_2 \geq t_1$ and proceed as in the proof of first case of Theorem 2.1. From (9), we have

$$w^{\Delta}(t) \leq -z(t)q(t) + A(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - z(t)\frac{1}{\kappa(z^{\sigma}(t))^2 r(t)}(w^{\sigma}(t))^2, \tag{16}$$

where

$$A(t) = z^{\Delta}(t) - \frac{z(t)}{r(t)}.$$

The proof is similar to that of Theorem 2.1 and hence is omitted.

**Corollary 3.2.** Assume that $(H_1)-(H_3)$ hold. If

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[ q(s) - \frac{\kappa}{4r(s)} \right] \Delta s = \infty \tag{17}$$

then equation (1) is oscillatory.

**Theorem 3.3.** Assume that $(H_1)-(H_3)$ hold.Let $z(t)$ be positive real rd-functions differentiable function and let $H : \mathbb{D} \to \mathbb{R}$ be rd-continuous function such that H belongs to the class $\mathfrak{R}$ . If

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[ H(t,s)z(s)q(s) - \frac{\kappa r(s)C^2(t,s)}{4z(s)H(t,s)} \left( z^{\Delta}(s) - \frac{z(s)}{r(s)} \right)^2 \right] \Delta s = \infty,$$

where

$$C(t,s) = z^{\Delta}(s)H_s^{\Delta}(t,s) + H(t,s),$$

then equation (1) is oscillatory.

**Corollary 2.4.** Assume that $(H_1)-(H_3)$ hold. Let $z(t) = 1$. If

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s) \left( q(s) - \frac{\kappa}{4r(s)} \right) \Delta s = \infty,$$

then every solution of (1) is oscillatory.

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## References

[1] S. Hilger, Analysis on measure chains a unified approach to continuous and discrete calculus, Results Math. 18 , 18-56, (1990).
[2] R.P. Agarwal, M. Bohner, D. O'Regan, A. Peterson, Dynamic equations on time scales: A survey, in: R.P. Agarwal, M. Bohner, D. O'Regan (Eds.), Special Issue on Dynamic Equations on Time Scales , J. Comput. Appl. Math. 141 (12) , 1-26, (2002).
[3] M. Bohner, A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser, Boston, 2001.
[4] R.P. Agarwal, M. Bohner, S.H. Saker, Oscillation criteria for nonlinear perturbed dynamic equations of second-order on time scales , J. Appl. Math. and Computing, 20, No.1-2, 133-147, (2006).
[5] Samir H. Saker, Ravi P. Agarwal, Donal O'Regan, Oscillation of second-order damped dynamic equations on time scales, J. Math. Anal. and App., 330 , 1317-1337, (2007).
[6] M. Bohner, L. Erbe, A. Peterson, Oscillation for second order dynamic equations on time scale, J. Math., Anal., App., 301 , 491-507, (2005) .
[7] L. Erbe, A. Peterson, Riccati equations on a measure chain, in: G.S. Ladde, N.G. Medhin, M. Sambandham (Eds.), Proceedings of Dynamic Systems and Applications, vol. 3, Dynamic publishers, Atlanta, pp. 193-199, (2001).
[8] S.R. Grace, R.P. Agarwal, B. Kaymakçalan, W. Sae-jie, Oscillation theorems for second order nonlinear dynamic equations, J. Appl. Math. Comput. 32 , 205-218, (2010).
[9] Taher S. Hassan, Lynn Erbe, Allan Peterson, Oscillation Theorems of Second Order Superlinear Dynamic Equations with Damping on Time Scales, Com. Math. Appl., 59 , 550-558, (2010).
[10] M. Bohner, S.H. Saker, Oscillation of second order nonlinear dynamic equations on time scales, Rocky Mountain J. Math. 34, 1239-1254, (2004).
[11] M. Bohner, S.H. Saker, Oscillation criteria for perturbed nonlinear dynamic equations, Math. Comput. Modelling, 40 , 249-260, (2004).
[12] L. Erbe, Oscillation criteria for second order linear equations on a time scale, Canad. Appl. Math. Quart., 9 , 1-31, (2001).
[13] S.R. Grace, Oscillation Theorems for nonlinear differential equations of second order, J. of Math., Anal., App., 171 , 220-241, (1992).
[14] L. Erbe, A. Peterson, Boundedness and oscillation for nonlinear dynamic equations on a time scale, Proc. Amer. Math. Soc., 132 , 735-744, (2004).
[15] L. Erbe, A. Peterson, S.H. Saker, Oscillation criteria for second-order nonlinear dynamic equations on time scales, J. London Math. Soc. 67, 701-714, (2003).
[16] L. Erbe, A. Peterson, S.H. Saker, Asymptotic behavior of solutions of a third-order nonlinear dynamic equation on time scales, J. Comput. Appl. Math. 181 , 92-102, (2005).
[17] S.H. Saker, Oscillation of nonlinear dynamic equations on time scales, Appl. Math. Comput. 148 , 81-91, (2004).
[18] S.H. Saker, Oscillation criteria of second-order half-linear dynamic equations on time scales, J. Comput. Appl. Math. 177 , 375-387, (2005).
[19] C.C. Yeh, Oscillation theorems for nonlinear second order differential equations with damped term, Proc. Amer. Math. Soc. 84 , 397-402, (1982).

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*Research Article*

# Kamenev-Type Oscillation Criteria for the Second-Order Nonlinear Dynamic Equations with Damping on Time Scales

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The oscillation of solutions of the second-order nonlinear dynamic equation $(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + p(t)(x^{\Delta}(t))^{\gamma} + f(t, x(g(t))) = 0$, with damping on an arbitrary time scale $\mathbb{T}$, is investigated. The generalized Riccati transformation is applied for the study of the Kamenev-type oscillation criteria for this nonlinear dynamic equation. Several new sufficient conditions for oscillatory solutions of this equation are obtained.

## 1. Introduction

Much recent attention has been given to dynamic equations on time scales, or measure chains, and we refer the reader to the landmark paper of Hilger [1] for a comprehensive treatment of the subject. Since then, several authors have expounded on various aspects of this new theory; see the survey paper by Agarwal et al. [2]. A book on the subject of time scales by Bohner and Peterson [3] also summarizes and organizes much of the time scale calculus.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The forward and the backward jump operators on any time scale $\mathbb{T}$ are defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$. A point $t \in \mathbb{T}$, $t > \inf \mathbb{T}$, is said to be left-dense if $\rho(t) = t$, right dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, left scattered if $\rho(t) < t$, and right scattered if $\sigma(t) > t$. The graininess function $\mu$ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$. For a function $f : \mathbb{T} \to \mathbb{R}$ the (delta) derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}, \tag{1.1}$$

if $f$ is continuous at $t$ and $t$ is right scattered. If $t$ is not right scattered, then the derivative is

defined by

$$f^{\Delta}(t) = \lim_{s \to t^{+}} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \lim_{s \to t^{+}} \frac{f(t) - f(s)}{t - s}, \tag{1.2}$$

provided this limit exists. A function $f : [a, b] \to \mathbb{R}$ is said to be right-dense continuous if it is right continuous at each right-dense point and there exists a finite left limit at all left-dense points, and $f$ is said to be differentiable if its derivative exists. A useful formula dealing with the time scale is that

$$f^{\sigma} = f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t). \tag{1.3}$$

We will make use of the following product and quotient rules for the derivative of the product $fg$ and the quotient $f/g$ (where $gg^{\sigma} \neq 0$) of two differentiable functions $f$ and $g$:

$$\begin{gathered} (fg)^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta} = f g^{\Delta} + f^{\Delta} g^{\sigma}, \\ \left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta} g - f g^{\Delta}}{g g^{\sigma}}. \end{gathered} \tag{1.4}$$

The integration by parts formula is

$$\int_a^b f^{\Delta}(t) g(t) \Delta t = f(b) g(b) - f(a) g(a) - \int_a^b f^{\sigma}(t) g^{\Delta}(t) \Delta(t). \tag{1.5}$$

The function $f : \mathbb{T} \to \mathbb{R}$ is called *rd*-continuous if it is continuous at the right-dense points and if the left-sided limits exist in left-dense points. We denote the set of all $f : \mathbb{T} \to \mathbb{R}$ which are *rd*-continuous and regressive by $\mathfrak{R}$. If $p \in \mathfrak{R}$, then we can define the exponential function by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(t)}(p(\tau(t))) \Delta \tau\right) \tag{1.6}$$

for $t \in \mathbb{T}$, $s \in \mathbb{T}^k$, where $\xi_h(z)$ is the cylinder transformation, which is defined by

$$\xi_h(z) = \begin{cases} \dfrac{\log(1 + hz)}{h}, & h \neq 0, \\ z, & h = 0. \end{cases} \tag{1.7}$$

Alternately, for $p \in \mathfrak{R}$ one can define the exponential function $e_p(\cdot, t_0)$, to be the unique solution of the IVP $x^{\Delta}(t) = p(t) x(t)$ with $x(t_0) = 1$.

The various-type oscillation and nonoscillation criteria for solutions of ordinary and partial differential equations have been studied extensively in a large cycle of works (see [4–31]).

In [27], the authors have considered second-order nonlinear neutral dynamic equation

$$\left(r(t)\left(\left(y(t)+p(t)y(t-\tau)\right)^{\Delta}\right)^{\gamma}\right)^{\Delta}+f\left(t,y(t-\delta)\right)=0 \tag{1.8}$$

on a time scale $\mathbb{T}$. They have assumed that $\gamma > 0$ is a quotient of odd positive integers, $\tau$ and $\delta$ positive constants such that the delay functions $\tau(t) = t-\tau < t$ and $\delta(t) = t-\delta < t$ satisfy $\tau(t)$ and $\delta(t) : \mathbb{T} \to \mathbb{T}$ for all $t \in \mathbb{T}$, $r(t)$ and $p(t)$ real-valued positive functions defined on $\mathbb{T}$ and also they have supposed that

(H1) $\int_{t_0}^{\infty}(1/r(t))^{1/\gamma}\Delta t = \infty,\ 0 \le p(t) < 1$,

(H2) $f(t,u) : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$ is continuous such that $uf(t,u) > 0$ for all $u \neq 0$ and there exists a nonnegative function $q(t)$ defined on $\mathbb{T}$ such that $|f(t,u)| \ge q(t)|u^{\gamma}|$

and were concerned with oscillation properties of (1.8). In [28], Saker has considered second-order nonlinear neutral delay dynamic equation

$$\left(r(t)\left(\left(y(t)+p(t)y(t-\tau)\right)^{\Delta}\right)^{\gamma}\right)^{\Delta}+f\left(t,y(t-\delta)\right)=0, \tag{1.9}$$

when $\gamma \ge 1$ is an odd positive integer with $r(t)$ and $p(t)$ real-valued positive functions defined on $\mathbb{T}$. The author also has improved some well-known oscillation results for second-order neutral delay difference equations. Agarwal et al. [29] have considered the second-order perturbed dynamic equation

$$\left(r(t)\left(x^{\Delta}\right)^{\gamma}\right)^{\Delta}+F(t,x(t))=G\left(t,x(t),x^{\Delta}(t)\right), \tag{1.10}$$

where $\gamma \in \mathbb{N}$ is odd and they have interested in asymptotic behavior of solutions of (1.10). Saker et al. [30] have studied the second-order damped dynamic equation with damping

$$\left(a(t)x^{\Delta}(t)\right)^{\Delta}+p(t)x^{\Delta^{\sigma}}(t)+q(t)\left(fox^{\sigma}\right)=0, \tag{1.11}$$

when $a(t)$, $p(t)$, and $q(t)$ are positive real-valued $rd$-continuous functions and they have proved that if $\int_{t_0}^{\infty}(e_{-p/r}(t,t_0)/r(t))\Delta t = \infty$ and $\int_{t_0}^{\infty}(e_{-p/r}(t,t_0)/r(t))\Delta t < \infty$, then every solution of (1.11) is oscillatory.

In the present paper, we consider the second order nonlinear dynamic equation

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+p(t)\left(x^{\Delta}(t)\right)^{\gamma}+f\left(t,x\left(g(t)\right)\right)=0, \tag{1.12}$$

where $p$, $r$ are real-valued, nonnegative, and right-dense continuous function on a time scale $\mathbb{T} \subset \mathbb{R}$, with $\sup \mathbb{T} = \infty$ and $\gamma$ is a quotient of odd positive integers. We assume that $g : \mathbb{T} \to \mathbb{T}$ is a nondecreasing function and such that $g(t) \ge t$, for $t \in \mathbb{T}$ and $\lim_{t\to\infty} g(t) = \infty$. The function $f \in C(\mathbb{T}\times\mathbb{R},\mathbb{R})$ is assumed to satisfy $uf(t,u) > 0$, for $u \neq 0$ and there exists a positive

$rd$-continuous function $q$ defined on $\mathbb{T}$ such that $|f(t,u)/u^{\gamma}| \geq q(t)$ for $u \neq 0$. Throughout this paper we assume that

$$\int_{t_0}^{\infty} \left( \frac{e_{-p/r}(t,t_0)}{r(t)} \right)^{1/\gamma} \Delta t = \infty. \tag{$A^*$}$$

Since we are interested in the oscillatory of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$ and define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}} := [t_0,\infty) \cap \mathbb{T}$. The oscillation of solutions of the second-order nonlinear dynamic equation (1.12) with damping on an arbitrary time scale $\mathbb{T}$ is investigated. The generalized Riccati transformation is applied for the study of the Kamenev-type oscillation criteria for this nonlinear dynamic differential equation. Several new sufficient conditions for oscillatory solutions of this equation are obtained.

A solution $x(t)$ of (1.12) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory.

## 2. Preliminary Results

**Lemma 2.1.** *Assume that the condition* $(A^*)$ *is satisfied and* (1.12) *has a positive solution* $x(t)$ *on* $[t_0,\infty)_{\mathbb{T}}$. *Then there exists a sufficiently large* $t_1 \in [t_0,\infty)_{\mathbb{T}}$ *such that*

$$\left( r(t) \left( x^{\Delta}(t) \right)^{\gamma} \right)^{\Delta} < 0, x^{\Delta}(t) > 0 \quad \text{for } t \in [t_1,\infty)_{\mathbb{T}}. \tag{2.1}$$

*Proof.* Let $t_1 \in [t_0,\infty)$ such that $x(g(t)) > 0$ on $[t_1,\infty)$. Since $x(t)$ is positive nonoscillatory solution of (1.12) we can assume that $x^{\Delta}(t) < 0$ for all large $t$. Then without loss of generality we take $x^{\Delta}(t) < 0$ for all $t \geq t_2 \geq t_1$. From (1.12) it follows that

$$\left( r(t) \left( x^{\Delta}(t) \right)^{\gamma} \right)^{\Delta} + p(t) \left( x^{\Delta}(t) \right)^{\gamma} = -f(t, x(g(t))) < 0 \tag{2.2}$$

and so

$$\left( r(t) \left( x^{\Delta}(t) \right)^{\gamma} \right)^{\Delta} + p(t) \left( x^{\Delta}(t) \right)^{\gamma} < 0. \tag{2.3}$$

Define $y(t) = -r(t)(x^{\Delta}(t))^{\gamma}$. Hence

$$y^{\Delta}(t) + \frac{p(t)}{r(t)} y(t) > 0, \tag{2.4}$$

and it implies that

$$y(t) > y(t_2) e_{-p/r}(\cdot, t_2). \tag{2.5}$$

Then

$$-r(t)\left(x^{\Delta}(t)\right)^{\gamma} > -r(t_2)\left(x^{\Delta}(t_2)\right)^{\gamma} e_{-p/r}(\cdot, t_2), \tag{2.6}$$

and therefore

$$x^{\Delta}(t) \le r^{1/\gamma}(t_2)\left(x^{\Delta}(t_2)\right)\left(\frac{e_{-p/r}(\cdot, t_2)}{r(t)}\right)^{1/\gamma}. \tag{2.7}$$

Next an integration for $t > t_3 \ge t_2$ and by $(A^*)$ gives

$$x(t) \le x(t_3) + r^{1/\gamma}(t_2)\left(x^{\Delta}(t_2)\right)\int_{t_3}^{t}\left(\frac{e_{-p/r}(s, t_2)}{r(s)}\right)^{1/\gamma}\Delta s \longrightarrow -\infty \quad \text{as } t \longrightarrow \infty \tag{2.8}$$

which is a contradiction. Hence $x^{\Delta}(t)$ is not negative for all large $t$ and so $x^{\Delta}(t) > 0$ for all $t \ge t_1$. This completes the proof of Lemma 2.1. □

We now define

$$\begin{aligned} \alpha_1(t) &:= \left(\frac{1}{r(t)}\int_t^{\infty} q(s)\Delta s\right)^{(1-\gamma)/\gamma} \\ \alpha_2(t,u) &:= \left(r^{1/\gamma}(t)\int_u^t \frac{\Delta s}{r^{1/\gamma}(s)}\right)^{\gamma-1} \\ \alpha(t) &:= \begin{cases} \alpha_1(t), & 0 < \gamma \le 1, \\ \alpha_2(t, t_1), & \gamma \ge 1. \end{cases} \end{aligned} \tag{2.9}$$

**Lemma 2.2.** *Assume that* $(A^*)$ *holds and* (1.12) *has a positive solution* $x(t)$ *on* $[t_0, \infty)_{\mathbb{T}}$. *Then there exists a sufficiently large* $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *such that if* $0 < \gamma \le 1$ *for* $t \ge t_1$ *one has*

$$\left(\frac{x^{\Delta}(t)}{x^{\sigma}(t)}\right)^{1-\gamma} \ge \alpha_1(t). \tag{2.10}$$

*Whereas, if* $\gamma \ge 1$, *one has*

$$\left(\frac{x(t)}{x^{\Delta}(t)}\right)^{\gamma-1} \ge \alpha_2(t, t_1) \quad \text{for } t \ge t_1. \tag{2.11}$$

*Proof.* As in the proof of Lemma 2.1, there is a sufficiently large $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$x(t) > 0, \quad x^{\Delta}(t) > 0, \quad \left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} < 0, \quad \text{for } t \ge t_1. \tag{2.12}$$

From (1.12) and (2.12) it follows that

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+p(t)\left(x^{\Delta}(t)\right)^{\gamma}=-f\left(t,x\left(g(t)\right)\right)<0, \tag{2.13}$$

and so

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}<-f\left(t,x\left(g(t)\right)\right). \tag{2.14}$$

Then

$$\begin{aligned} r(t)\left(x^{\Delta}(t)\right)^{\gamma} &\geq \int_{t}^{\infty} f\left(s,x\left(g(s)\right)\right)\Delta s \geq \int_{t}^{\infty} q(s)x^{\gamma}\left(g(s)\right)\Delta s \\ &\geq x^{\gamma}\left(g(t)\right)\int_{t}^{\infty} q(s)\Delta s \geq \left(x^{\sigma}(t)\right)^{\gamma}\int_{t}^{\infty} q(s)\Delta s. \end{aligned} \tag{2.15}$$

Next, when $0<\gamma\leq 1$, we get

$$\left(\frac{x^{\Delta}(t)}{x^{\sigma}(t)}\right)^{1-\gamma}\geq \alpha_1(t) \quad \text{for } t\geq t_1. \tag{2.16}$$

Finally, since $r(t)(x^{\Delta}(t))^{\gamma}$ is decreasing on $[t_1,\infty)_{\mathbb{T}}$ for $\gamma\geq 1$, we get

$$\begin{aligned} x(t)\geq x(t)-x(t_1) &= \int_{t_1}^{t}\frac{\left(r(s)\left(x^{\Delta}(s)\right)^{\gamma}\right)^{1/\gamma}}{r^{1/\gamma}(s)}\Delta s \\ &\geq \left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{1/\gamma}\int_{t_1}^{t}\frac{1}{r^{1/\gamma}(s)}\Delta s, \end{aligned} \tag{2.17}$$

and we obtain

$$\left(\frac{x(t)}{x^{\Delta}(t)}\right)^{\gamma-1}\geq \alpha_2(t,t_1) \quad \text{for } t\geq t_1. \tag{2.18}$$

□

## 3. Main Results

**Theorem 3.1.** *Assume that* $(A^*)$ *holds and there exist a function* $\phi(t)$ *such that* $r(t)\phi(t)$ *is a* $\Delta$*-differentiable function and a positive real rd-functions* $\Delta$*-differentiable function* $z(t)$ *such that*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\Psi(s)-\frac{1}{4}\frac{r(s)(\nu(s))^2}{\gamma z(s)\alpha(s)}\right]\Delta s=\infty, \tag{3.1}$$

*where*

$$\Psi(t) = -z(t)\left(q(s) - \left(r(t)\phi(t)\right)^{\Delta} + \frac{\gamma\alpha(t)}{r(t)}\left(p(t)\left(r(t)\phi(t)\right)^{\sigma} + \left(\left(r(t)\phi(t)\right)^{\sigma}\right)^{2}\right)\right),$$
$$\nu(t) = z^{\Delta}(t) - \frac{\gamma z(t)\alpha(t)}{r(t)}\left(p(t) - 2\left(r(t)\phi(t)\right)^{\sigma}\right). \tag{3.2}$$

*Then every solution of* (1.12) *is oscillatory.*

*Proof.* Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (1.12). Without loss of generality, there is a $t_1 \in [t_0, \infty)_{\mathbb{T}}$, sufficiently large, so that $x(t)$ satisfies the conclusions of Lemmas 2.1 and 2.2 on $[t_0, \infty)_{\mathbb{T}}$. Define the function $w(t)$ by Riccati substitution

$$w(t) = z(t)r(t)\left(\left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\gamma} + \phi(t)\right), \quad t \geq t_1. \tag{3.3}$$

Then $w(t)$ satisfies

$$\begin{aligned} w^{\Delta}(t) &= \left(\frac{z(t)}{x^{\gamma}(t)}\right)\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + \left(\frac{z(t)}{x^{\gamma}(t)}\right)^{\Delta}\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma} \\ &\quad + z(t)\left(r(t)\phi(t)\right)^{\Delta} + z^{\Delta}(t)\left(r(t)\phi(t)\right)^{\sigma}, \\ w^{\Delta}(t) &= \left(\frac{z(t)}{x^{\gamma}(t)}\right)\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + \left(\frac{z^{\Delta}(t)x^{\gamma}(t) - z(t)\left(x^{\gamma}(t)\right)^{\Delta}}{x^{\gamma}(t)\left(x^{\gamma}(t)\right)^{\sigma}}\right)\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma} \\ &\quad + z(t)\left(r(t)\phi(t)\right)^{\Delta} + z^{\Delta}(t)\left(r(t)\phi(t)\right)^{\sigma}. \end{aligned} \tag{3.4}$$

From (1.12) and the definition of $w(t)$ for $t \geq t_1$ it follows that

$$\begin{aligned} w^{\Delta}(t) &= \left(\frac{z(t)}{x^{\gamma}(t)}\right)\left(-p(t)\left(x^{\Delta}(t)\right)^{\gamma} - f\left(t, x\left(g(t)\right)\right)\right) + z^{\Delta}(t)\frac{\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}}{\left(x^{\gamma}(t)\right)^{\sigma}} \\ &\quad - z(t)\frac{\left(x^{\gamma}(t)\right)^{\Delta}\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}}{x^{\gamma}(t)\left(x^{\gamma}(t)\right)^{\sigma}} + z(t)\left(r(t)\phi(t)\right)^{\Delta} + z^{\Delta}(t)\left(r(t)\phi(t)\right)^{\sigma}. \end{aligned} \tag{3.5}$$

Using the fact that $f(t, x(g(t))) \geq q(t)x^{\gamma}(g(t))$ and $x(t)$ is a increasing function, we obtain

$$\begin{aligned} w^{\Delta}(t) &\leq -z(t)q(t) - z(t)p(t)\frac{\left(x^{\gamma}(t)\right)^{\Delta}}{x^{\gamma}(t)} + z^{\Delta}(t)\left(\left(\frac{r(t)\left(x^{\Delta}(t)\right)^{\gamma}}{x^{\gamma}(t)}\right)^{\sigma} + \left(r(t)\phi(t)\right)^{\sigma}\right) \\ &\quad - z(t)\frac{\left(x^{\Delta}(t)\right)^{\gamma}}{x^{\gamma}(t)}\left(\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \left(r(t)\phi(t)\right)^{\sigma}\right) + z(t)\left(r(t)\phi(t)\right)^{\Delta}. \end{aligned} \tag{3.6}$$

Now we consider the following two cases: $0 < \gamma \leq 1$ and $\gamma > 1$.

In the first case $0 < \gamma \leq 1$. Using the Pötzsche chain rule (see, [3]), we obtain

$$(x^{\gamma}(t))^{\Delta} = \gamma \int_0^1 \left[x(t) + h\mu(t)x^{\Delta}(t)\right]^{\gamma-1} dh\, x^{\Delta}(t) \geq \gamma (x^{\sigma}(t))^{\gamma-1} x^{\Delta}(t). \tag{3.7}$$

Using (3.7) in (3.6) for $t \geq t_1$, we get

$$\begin{aligned} w^{\Delta}(t) \leq{}& -z(t)q(t) - \gamma z(t)p(t)\frac{x^{\Delta}(t)}{x^{\sigma}(t)}\left(\frac{x^{\sigma}(t)}{x(t)}\right)^{\gamma} + z^{\Delta}(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} \\ &- \gamma z(t)\frac{x^{\Delta}(t)}{x^{\sigma}(t)}\left(\frac{x^{\sigma}(t)}{x(t)}\right)^{\gamma}\left(\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \left(r(t)\phi(t)\right)^{\sigma}\right) + z(t)\left(r(t)\phi(t)\right)^{\Delta}. \end{aligned} \tag{3.8}$$

By Lemmas 2.1 and 2.2, for $t \geq t_1$, we have that

$$\begin{gathered} \frac{x^{\Delta}(t)}{x^{\sigma}(t)} = \frac{1}{r(t)}\frac{r(t)\left(x^{\Delta}(t)\right)^{\gamma}}{(x^{\gamma}(t))^{\sigma}}\left(\frac{x^{\Delta}(t)}{x^{\sigma}(t)}\right)^{1-\gamma} \geq \frac{\alpha_1(t)}{r(t)}\frac{\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}}{(x^{\gamma}(t))^{\sigma}}, \\ \frac{x^{\sigma}(t)}{x(t)} \geq 1. \end{gathered} \tag{3.9}$$

In the view of (3.8), and (3.9) we get

$$\begin{aligned} w^{\Delta}(t) \leq{}& -z(t)q(t) + z(t)\left(r(t)\phi(t)\right)^{\Delta} - \gamma z(t)p(t)\frac{\alpha_1(t)}{r(t)}\left(\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \left(r(t)\phi(t)\right)^{\sigma}\right) \\ &+ z^{\Delta}(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \gamma z(t)\frac{\alpha_1(t)}{r(t)}\left(\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \left(r(t)\phi(t)\right)^{\sigma}\right)^2. \end{aligned} \tag{3.10}$$

In the second case $\gamma > 1$. Applying the Pötzsche chain rule (see, [3]), we obtain

$$(x^{\gamma}(t))^{\Delta} = \gamma \int_0^1 \left[x(t) + h\mu(t)x^{\Delta}(t)\right]^{\gamma-1} dh\, x^{\Delta}(t) \geq \gamma (x(t))^{\gamma-1} x^{\Delta}(t). \tag{3.11}$$

In the view of (3.11), (3.6) yields

$$\begin{aligned} w^{\Delta}(t) \leq{}& -z(t)q(t) + z(t)\left(r(t)\phi(t)\right)^{\Delta} - \gamma z(t)p(t)\frac{(x(t))^{\gamma-1}}{x^{\gamma}(t)}x^{\Delta}(t) \\ &+ z^{\Delta}(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \gamma z(t)\frac{(x(t))^{\gamma-1}}{x^{\gamma}(t)}x^{\Delta}(t)\left(\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \left(r(t)\phi(t)\right)^{\sigma}\right). \end{aligned} \tag{3.12}$$

By Lemmas 2.1 and 2.2, we have that

$$\frac{x^{\Delta}(t)}{x(t)} = \frac{1}{r(t)}\frac{r(t)\left(x^{\Delta}(t)\right)^{\gamma}}{x^{\gamma}(t)}\left(\frac{x(t)}{x^{\Delta}(t)}\right)^{\gamma-1} \geq \frac{\alpha_2(t,t_1)}{r(t)}\frac{\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}}{(x^{\gamma}(t))^{\sigma}}. \tag{3.13}$$

By (3.13), (3.12), and then using the definition of $w(t)$, we get

$$\begin{aligned} w^{\Delta}(t) \leq & -z(t)q(t) + z(t)\left(r(t)\phi(t)\right)^{\Delta} - \gamma z(t)p(t)\frac{\alpha_2(t,t_1)}{r(t)}\left(\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \left(r(t)\phi(t)\right)^{\sigma}\right) \\ & + z^{\Delta}(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \gamma z(t)\frac{\alpha_2(t,t_1)}{r(t)}\left(\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \left(r(t)\phi(t)\right)^{\sigma}\right)^2. \end{aligned} \tag{3.14}$$

Using (3.10), (3.14), and the definitions of $\Psi(t)$, $\nu(t)$, and $\alpha(t)$ for $\gamma > 0$, we get

$$w^{\Delta}(t) \leq -\Psi(t) + \nu(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \gamma z(t)\frac{\alpha(t)}{r(t)}\frac{\left(w^{\sigma}(t)\right)^2}{\left(z^{\sigma}(t)\right)^2}. \tag{3.15}$$

Then, we can write

$$w^{\Delta}(t) \leq -\Psi(t) + \frac{r(t)(\nu(t))^2}{4\gamma z(t)\alpha(t)} - \left[\sqrt{\frac{\gamma z(t)\alpha(t)}{r(t)}}\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \frac{1}{2}\sqrt{\frac{r(t)}{\gamma z(t)\alpha(t)}}\nu(t)\right]^2, \tag{3.16}$$

and so, we get

$$w^{\Delta}(t) \leq -\left[\Psi(t) - \frac{r(t)(\nu(t))^2}{4\gamma z(t)\alpha(t)}\right]. \tag{3.17}$$

Integrating (3.17) with respect to $s$ from $t_1$ to $t$, we get

$$w(t) - w(t_1) \leq -\int_{t_1}^{t}\left[\Psi(s) - \frac{r(s)(\nu(s))^2}{4\gamma z(s)\alpha(s)}\right]\Delta s, \tag{3.18}$$

and this implies that

$$\int_{t_1}^{t}\left[\Psi(s) - \frac{r(s)(\nu(s))^2}{4\gamma z(s)\alpha(s)}\right]\Delta s \leq |w(t_1)| \tag{3.19}$$

which contradicts to assumption (3.1). This completes the proof of Theorem 3.1. □

**Corollary 3.2.** *Assume that* $(A^*)$ *holds. If*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[q(s) + \frac{\gamma\alpha(s)p^2(s)}{4r(s)}\right]\Delta s = \infty, \tag{3.20}$$

*then every solution of* (1.12) *is oscillatory.*

*Example 3.3.* Consider the nonlinear dynamic equation

$$\left(t^{-\gamma}\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+\frac{1}{2}t^{-1-\gamma}\left(x^{\Delta}(t)\right)^{\gamma}+\frac{1}{t^{1/\gamma}}x^{\gamma}(g(t))=0,\quad t\in[t_0,\infty)_{\mathbb{T}},\ \mathbb{T}=2^{\mathbb{N}},\tag{3.21}$$

where $\gamma \geq 1$ is the quotient of the odd positive integers. We have that $p(t) = (1/2)(t^{-1-\gamma}), q(t) = 1/t^{1/\gamma}$ and $r(t) = t^{-\gamma}$. If $\mathbb{T} = 2^{\mathbb{N}}$, then $\sigma(t) = 2t$ and $e_{-1/\sigma(t)}(t, t_0) = t_0/t$. So we get $e_{-p/r}(t, t_0) = t_0/t$. It is clear that $(A^*)$ holds. Indeed,

$$\begin{gathered}\int_{t_0}^{t}\left(\frac{e_{-p/r}(\cdot,t_0)}{r(s)}\right)^{1/\gamma}\Delta s=(t_0)^{1/\gamma}\int_{t_0}^{t}\frac{1}{s^{(1/\gamma)-1}}\Delta s=\infty,\\ \alpha_2(t,t_0)=\left((r(t))^{1/\gamma}\int_{t_0}^{t}\frac{\Delta s}{(r(s))^{1/\gamma}}\right)^{\gamma-1}=t^{1/(\gamma-1)}\left(\int_{t_0}^{t}\frac{\Delta s}{s^{-1}}\right)^{\gamma-1},\end{gathered}\tag{3.22}$$

and then

$$\int_{t_0}^{t}\frac{\Delta s}{s^{-1}}=\infty\tag{3.23}$$

and so we can find $t_* \geq t_1$ such that $\int_{t_0}^{t} \Delta s/r^{1/\gamma} \geq 1$ for $t \geq t_*$. Then we can see from Corollary 3.2 that it follows that

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\frac{1}{s^{1/\gamma}}+\frac{\gamma\alpha(s)(p(s))^2}{4r(s)}\right]\Delta s=\infty,\tag{3.24}$$

and therefore every solution of (3.21) is oscillatory.

Now, let us introduce the class of functions $\mathfrak{R}$.

Let $\mathbb{D}_0 \equiv \{(t,s) \in \mathbb{T}^2 : t > s \geq t_0\}$ and $\mathbb{D} \equiv \{(t,s) \in \mathbb{T}^2 : t \geq s \geq t_0\}$. The function $H \in C_{rd}(\mathbb{D}, \mathbb{R})$ has the following properties:

$$H(t,t)=0,\quad t\geq t_0,\quad H(t,s)>0,\quad \text{on } \mathbb{D}_0,\tag{3.25}$$

and $H$ has a continuous $\Delta$-partial derivative $H_s^{\Delta}(t,s)$ on $\mathbb{D}_0$ with respect to the second variable. ($H$ is $rd$-continuous function if $H$ is $rd$-continuous function in $t$ and $s$.)

**Theorem 3.4.** *Assume that the conditions of Lemma 2.1 are satisfied. Furthermore, suppose that there exist functions $H, H_s^{\Delta} \in C_{rd}(\mathbb{D}, \mathbb{R})$ such that* (3.25) *holds and there exist a function $\phi(t)$ with $r(t)\phi(t)$ a $\Delta$-differentiable function and a positive $\Delta$-differentiable function $z(t)$ such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)\Psi(s)-\frac{r(s)}{4\gamma H(t,s)z(s)\alpha(s)}\varphi^2(t,s)\right]\Delta s=\infty,\tag{3.26}$$

*where $\varphi(t,s) = [H_s^{\Delta}(t,s) + H(t,s)\nu(s)]$. Then every solution of* (1.12) *is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.*

*Proof.* Assume that (1.12) has a nonoscillatory solution on $[t_0,\infty)_{\mathbb{T}}$. Then without loss of generality, there is a sufficiently large $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t)$ satisfies the conclusions of Lemmas 2.1 and 2.2 on $[t_0,\infty)_{\mathbb{T}}$. Consider the generalized Riccati substitution

$$w(t) = z(t)r(t)\left(\left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\gamma} + \phi(t)\right). \tag{3.27}$$

We proceed as Theorem 3.1 and from (3.15) it follows that

$$w^{\Delta}(t) \le -\Psi(t) + \nu(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \gamma z(t)\frac{\alpha(t)}{r(t)}\frac{(w^{\sigma}(t))^2}{(z^{\sigma}(t))^2}. \tag{3.28}$$

Multiplying both sides of (3.28) by $H(t,s)$ and integrating with respect to $s$ from $t_1$ to $t$ ($t \ge t_1$), we obtain

$$\begin{aligned}\int_{t_1}^{t} H(t,s)\Psi(s)\Delta(s) \le &-\int_{t_1}^{t} H(t,s)w^{\Delta}(s) + \int_{t_1}^{t} H(t,s)\nu(s)\frac{w^{\sigma}(s)}{z^{\sigma}(s)}\Delta s \\ &-\int_{t_1}^{t} \gamma H(t,s)z(s)\frac{\alpha(s)}{r(s)}\frac{(w^{\sigma}(s))^2}{(z^{\sigma}(s))^2}\Delta s.\end{aligned} \tag{3.29}$$

Integrating by parts, we get

$$\begin{aligned}\int_{t_1}^{t} H(t,s)\Psi(s)\Delta(s) \le{}& H(t,t_1)w(t_1) + \int_{t_1}^{t} H_s^{\Delta}(t,s)w^{\sigma}(s)\Delta s + \int_{t_1}^{t} H(t,s)\nu(s)\frac{w^{\sigma}(s)}{z^{\sigma}(s)}\Delta s \\ &-\int_{t_1}^{t} \gamma H(t,s)z(s)\frac{\alpha(s)}{r(s)}\frac{(w^{\sigma}(s))^2}{(z^{\sigma}(s))^2}\Delta s, \\ \int_{t_1}^{t} H(t,s)\Psi(s)\Delta(s) \le{}& H(t,t_1)w(t_1) + \int_{t_1}^{t} \left[H_s^{\Delta}(t,s) + H(t,s)\nu(s)\right]\frac{w^{\sigma}(s)}{z^{\sigma}(s)}\Delta s \\ &-\int_{t_1}^{t} \gamma H(t,s)z(s)\frac{\alpha(s)}{r(s)}\frac{(w^{\sigma}(s))^2}{(z^{\sigma}(s))^2}\Delta s.\end{aligned} \tag{3.30}$$

It is easy to see that

$$\begin{aligned}\int_{t_1}^{t} H(t,s)\Psi(s)\Delta(s) \le{}& H(t,t_1)w(t_1) + \int_{t_1}^{t} \varphi(t,s)\frac{w^{\sigma}(s)}{z^{\sigma}(s)}\Delta s \\ &-\int_{t_1}^{t} \gamma H(t,s)z(s)\frac{\alpha(s)}{r(s)}\frac{(w^{\sigma}(s))^2}{(z^{\sigma}(s))^2}\Delta s,\end{aligned} \tag{3.31}$$

where

$$\varphi(t,s) = \left[H_s^{\Delta}(t,s) + H(t,s)\nu(s)\right]. \tag{3.32}$$

Then we can write

$$\int_{t_1}^{t} H(t,s)\Psi(s)\Delta(s) \le H(t,t_1)w(t_1) + \int_{t_1}^{t} \frac{r(s)\varphi^2(t,s)}{4\gamma H(t,s)z(s)\alpha(s)}\Delta s - \int_{t_1}^{t}\left[\sqrt{\frac{\gamma H(t,s)z(s)\alpha(s)}{r(s)}}\frac{w^{\sigma}(s)}{z^{\sigma}(s)} - \frac{1}{2}\sqrt{\frac{r(s)}{\gamma H(t,s)z(s)\alpha(s)}}\varphi(t,s)\right]^2 \Delta s. \tag{3.33}$$

Hence

$$\begin{gathered}\int_{t_1}^{t} H(t,s)\Psi(s) - \frac{r(s)\varphi^2(t,s)}{4\gamma H(t,s)z(s)\alpha(s)}\Delta s \le H(t,t_1)w(t_1) \\ \limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)\Psi(s) - \frac{r(s)\varphi^2(t,s)}{4\gamma H(t,s)z(s)\alpha(s)}\right]\Delta s \le w(t_1)\end{gathered} \tag{3.34}$$

which contradicts with assumption (3.26). This completes the proof of Theorem 3.4. □

**Corollary 3.5.** *Assume that* $(A^*)$ *holds. Furthermore, suppose that there exist functions* $H, H_s^{\Delta}$, *and* $h \in C_{rd}(\mathbb{D},\mathbb{R})$ *such that* (3.25) *holds and there exist a function* $\phi(t)$ *such that* $r(t)\phi(t)$ *is a* $\Delta$-*differentiable function and a positive* $\Delta$-*differentiable function* $z(t)$ *such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)\Psi(s) - \frac{h^2(s)(z^{\sigma}(s))^2 r(s)}{4\gamma z(s)\alpha(s)}\right]\Delta s = \infty, \tag{3.35}$$

*where* $\Psi(t)$ *is as defined in Theorem 3.1 and* $H_s^{\Delta} = -h(t,s)\sqrt{H(t,s)} - H(t,s)\nu(t)/z^{\sigma}(t)$. *Then every solution of* (1.12) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

**Theorem 3.6.** *Assume that* $(A^*)$ *holds and there exists a* $\Delta$-*differentiable positive function* $z(t)$ *such that*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[z(s)q(s) - \frac{r(s)\xi^{\gamma+1}(s)}{(\gamma+1)^{\gamma+1}z^{\gamma}(s)}\right]\Delta s = \infty, \tag{3.36}$$

*where*

$$\xi(t) = z^{\Delta}(t) - \frac{z(t)p(t)}{r(t)}. \tag{3.37}$$

*Then every solution of* (1.12) *is oscillatory.*

*Proof.* Suppose that (1.12) has a nonoscillatory solution on $[t_0, \infty)_{\mathbb{T}}$. Then without loss of generality, there is a sufficiently large $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t)$ satisfies the conclusions of Lemmas 2.1 and 2.2 on $[t_0, \infty)_{\mathbb{T}}$. Consider the generalized Riccati substitution

$$w(t) = z(t)r(t)\left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\gamma}. \tag{3.38}$$

From (3.6) it follows that

$$w^{\Delta}(t) \leq -z(t)q(t) - z(t)p(t)\frac{\left(x^{\Delta}(t)\right)^{\gamma}}{x^{\gamma}(t)} + z^{\Delta}(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - z(t)\frac{\left(x^{\Delta}(t)\right)^{\gamma}}{x^{\gamma}(t)}\frac{w^{\sigma}(t)}{z^{\sigma}(t)}. \tag{3.39}$$

In the same manner as in the proof of Theorem 3.1, we get

$$\left(x^{\gamma}(t)\right)^{\Delta} \geq \begin{cases} \gamma(x^{\sigma}(t))^{\gamma-1}x^{\Delta}, & 0 < \gamma \leq 1 \\ \gamma(x(t))^{\gamma-1}x^{\Delta}, & \gamma > 1. \end{cases} \tag{3.40}$$

If $0 < \gamma \leq 1$, then we have that

$$w^{\Delta}(t) \leq -z(t)q(t) + \left[z^{\Delta}(t) - \frac{z(t)p(t)}{r(t)}\right]\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \gamma z(t)\frac{(x^{\sigma}(t))^{\gamma}}{x^{\gamma}(t)}\frac{x^{\Delta}(t)}{x^{\sigma}(t)}\frac{w^{\sigma}(t)}{z^{\sigma}(t)}, \tag{3.41}$$

whereas, if $\gamma > 1$, we have that

$$w^{\Delta}(t) \leq -z(t)q(t) + \left[z^{\Delta}(t) - \frac{z(t)p(t)}{r(t)}\right]\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \gamma z(t)\frac{x^{\sigma}(t)}{x(t)}\frac{x^{\Delta}(t)}{x^{\sigma}(t)}\frac{w^{\sigma}(t)}{z^{\sigma}(t)}. \tag{3.42}$$

Using the fact that $x(t)$ is increasing and $(r(t)(x^{\Delta}(t))^{\gamma}$ is decreasing on $[t_0, \infty)_{\mathbb{T}}$, we get

$$x^{\sigma}(t) \geq x(t), \qquad x^{\Delta}(t) \geq \left(\frac{r^{\sigma}(t)}{r(t)}\right)^{1/\gamma}\left(x^{\Delta}(t)\right)^{\sigma}. \tag{3.43}$$

Using (3.41), (3.42), and (3.43), we obtain

$$w^{\Delta}(t) \leq -z(t)q(t) + \xi(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - z(t)\frac{\gamma}{r^{1/\gamma}(t)}\left(\frac{w^{\sigma}(t)}{z^{\sigma}(t)}\right)^{\lambda}, \tag{3.44}$$

where $\lambda = (\gamma + 1)/\gamma$. Define $A > 0$ and $B > 0$ by

$$A^{\lambda} = \frac{\gamma z(t)(w^{\sigma}(t))^{\lambda}}{(z^{\sigma}(t))^{\lambda}r^{1/\gamma}(t)}, \qquad B^{\lambda-1} = \frac{r^{1/(\gamma+1)}(t)\xi(t)}{\lambda\gamma^{1/\lambda}z^{1/\lambda}(t)}. \tag{3.45}$$

Then using the inequality (see [32])

$$\lambda AB^{\lambda-1} - A^{\lambda} \leq (\lambda - 1)B^{\lambda}, \tag{3.46}$$

we obtain

$$\xi(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - z(t)\frac{\gamma}{r^{1/\gamma}(t)}\left(\frac{w^{\sigma}(t)}{z^{\sigma}(t)}\right)^{\lambda} \leq \frac{r(t)\xi^{\gamma+1}(t)}{(\gamma+1)^{\gamma+1}z^{\gamma}(t)}. \tag{3.47}$$

From this last inequality and (3.44) it follows that

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[z(s)q(s) - \frac{r(s)\xi^{\gamma+1}(s)}{(\gamma+1)^{\gamma+1}z^{\gamma}(s)}\right]\Delta s \leq w(t_1) \tag{3.48}$$

which contradicts with the assumption (3.36). Theorem 3.6 is proved. □

*Example 3.7.* Consider the second-order equation

$$\left(t^{\gamma}\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + \frac{1}{t^2}\left(x^{\Delta}(t)\right)^{\gamma} + \frac{1}{t}x^{\gamma}(g(t)) = 0, \tag{3.49}$$

where $\gamma = 1/3 \leq 1,\ r(t) = t^{1/3}, q(t) = 1/t,\ t \geq t_0 = 2$. Then it follows that

$$e_{-p/r}(t,2) \geq 1 - \int_2^t \frac{p(s)}{r(s)}\Delta s = 1 - \int_2^t s^{-7/3}\Delta s > \frac{1}{2} \tag{3.50}$$

for $t \geq 2$, and so

$$\int_2^t \left(\frac{1}{r(s)}e_{-p/r}(s,2)\right)^{1/\gamma}\Delta s \geq \left(\frac{1}{2}\right)^3\int_2^t \frac{1}{s}\Delta s \longrightarrow \infty \text{ as } t \longrightarrow \infty. \tag{3.51}$$

Hence $(A^*)$ is satisfied. Now let $z(t) = 1$ for $t \geq 2$. Then

$$\limsup_{t\to\infty}\int_2^t\left[z(s)q(s) - \frac{r(s)\xi^{\gamma+1}(s)}{(\gamma+1)^{\gamma+1}z^{\gamma}(s)}\right]\Delta s = \limsup_{t\to\infty}\int_2^t\left[\frac{1}{s} - \frac{s^{-25/9}}{(4/3)^{4/3}}\right]\Delta s = \infty, \tag{3.52}$$

and so (3.36) is satisfied as well. Hence by Theorem 3.6, we have that (3.49) is oscillatory.

**Theorem 3.8.** *Assume that the conditions of Lemma 2.1 hold. Furthermore, suppose that there exist functions $H, H_s^{\Delta} \in C_{rd}(\mathbb{D},\mathbb{R})$ such that (3.25) holds and there exists a positive real rd-functions $\Delta$-differentiable function $z(t)$ such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)z(s)q(s) - \frac{C^{\gamma+1}(t,s)r(s)}{(\gamma+1)^{\gamma+1}z^{\gamma}(s)(H(t,s))^{\gamma}}\right]\Delta s = \infty, \tag{3.53}$$

*where* $C(t,s) = H_s^{\Delta} z^{\sigma}(s) + H(t,s)\xi(t)$ *and* $\xi(t) = z^{\Delta}(t) - z(t)(p(t)/r(t))$. *Then every solution of* (1.12) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

*Proof*. Assume that (1.12) has a nonoscillatory solution on $[t_0,\infty)_{\mathbb{T}}$. Then without loss of generality, there is a sufficiently large $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t)$ satisfies the conclusions of Lemmas 2.1 and 2.2 on $[t_0,\infty)_{\mathbb{T}}$. Consider the generalized Riccati substitution

$$w(t) = z(t)r(t)\left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\gamma}. \tag{3.54}$$

By Theorem 3.6 and inequality (3.44)

$$w^{\Delta}(t) \le -z(t)q(t) + \xi(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - z(t)\frac{\gamma}{r^{1/\gamma}(t)}\left(\frac{w^{\sigma}(t)}{z^{\sigma}(t)}\right)^{\lambda}, \tag{3.55}$$

where $\lambda = (\gamma+1)/\gamma$. Multiplying both sides of (3.55) with $H(t,s)$ and integrating with respect to $s$ from $t_1$ to $t$ $(t \ge t_1)$, we get

$$\begin{aligned}\int_{t_1}^{t} H(t,s)z(s)q(s)\Delta s \le &-\int_{t_1}^{t} H(t,s)w^{\Delta}(s)\Delta(s) + \int_{t_1}^{t} H(t,s)\xi(s)\frac{w^{\sigma}(s)}{z^{\sigma}(s)} \\ &-\int_{t_1}^{t} H(t,s)z(s)\frac{\gamma}{r^{1/\gamma}(s)}\left(\frac{w^{\sigma}(s)}{z^{\sigma}(s)}\right)^{\lambda}\Delta s.\end{aligned} \tag{3.56}$$

Integrating by parts and using (3.25), we obtain

$$\int_{t_1}^{t} H(t,s)z(s)q(s)\Delta s \le H(t,t_1)w(t_1)\int_{t_1}^{t} C(t,s)\frac{w^{\sigma}(s)}{z^{\sigma}(s)} - \int_{t_1}^{t}\frac{\gamma H(t,s)z(s)}{r^{1/\gamma}(s)}\left(\frac{w^{\sigma}(s)}{z^{\sigma}(s)}\right)^{\lambda}\Delta s. \tag{3.57}$$

Define $A > 0$ and $B > 0$ by

$$A^{\lambda} = \frac{\gamma H(t,s)z(t)(w^{\sigma}(t))^{\lambda}}{(z^{\sigma}(t))^{\lambda} r^{1/\gamma}(t)}, \qquad B^{\lambda-1} = \frac{r^{1/(\gamma+1)}(t)C(t,s)}{\lambda\left(\gamma H(t,s)z(s)\right)^{1/\lambda}}. \tag{3.58}$$

Using the inequality (see [32])

$$\lambda A B^{\lambda-1} - A^{\lambda} \le (\lambda-1)B^{\lambda}, \tag{3.59}$$

we get

$$C(t,s)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \frac{\gamma H(t,s)z(t)}{r^{1/\gamma}(t)}\left(\frac{w^{\sigma}(t)}{z^{\sigma}(t)}\right)^{\lambda} \le \frac{r(t)C^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}H^{\gamma}(t,s)z^{\gamma}(t)}. \tag{3.60}$$

From this last inequality and (3.55) it follows that

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)z(s)q(s)-\frac{r(s)C^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}H^{\gamma}(t,s)z^{\gamma}(t)}\right]\Delta s\le w(t_1) \tag{3.61}$$

which contradicts with the assumption (3.53). This completes the proof of Theorem 3.8. □

**Corollary 3.9.** *Assume that all conditions of Lemma 2.1 hold. Furthermore, suppose that there exist functions $H, H_s^{\Delta}$, and $h\in C_{rd}(\mathbb{D},\mathbb{R})$ such that* (3.25) *holds and there exists a positive $\Delta$-differentiable function $z(t)$ such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)z(s)q(s)-\frac{(-h(t,s))^{\gamma+1}r(s)}{(\gamma+1)^{\gamma+1}z^{\gamma}(s)}\right]\Delta s=\infty, \tag{3.62}$$

*where $H_s^{\Delta}+H(t,s)\xi(t)/z^{\sigma}(s)=-h(t,s)(H(t,s))^{\gamma/(\gamma+1)}/z^{\sigma}(t)$. Then every solution of* (1.12) *is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.*

*Example 3.10.* Consider the second-order dynamic equation

$$\left(t^{\gamma}\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+\frac{1}{t^2}\left(x^{\Delta}(t)\right)^{\gamma}+\frac{1}{t}x^{\gamma}(g(t))=0, \tag{3.63}$$

where $t\in[t_0,\infty)_{\mathbb{T}}$, $t_1\ge t_0=2$, $\gamma=5/3\ge 1, q(t)=1/t$. It is easy to check that $(A^*)$ holds. For $z(t)=1$ and $H(t,s)=(t-s)^2$, it immediately follows that

$$h(t,s)=\left\{(t-s)-(t-s)^2+(t-\sigma(s))\right\}(t-s)^{2\gamma/(\gamma+1)} \tag{3.64}$$

and so $-h(t,s)=0$. Hence,

$$\limsup_{t\to\infty}\frac{1}{H(t,2)}\int_{2}^{t}\left[H(t,s)z(s)q(s)-\frac{(-h(t,s))^{\gamma+1}r(s)}{(\gamma+1)^{\gamma+1}z^{\gamma}(s)}\right]\Delta s=\limsup_{t\to\infty}\frac{1}{t^2}\int_{2}^{t}\frac{1}{s}(t-s)^2\Delta s=\infty. \tag{3.65}$$

Therefore by Corollary 3.9, every solution of (3.63) is oscillatory.

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## References


[1] S. Hilger, "Analysis on measure chains—a unified approach to continuous and discrete calculus," *Results in Mathematics*, vol. 18, no. 1-2, pp. 18–56, 1990.

[2] R. P. Agarwal, M. Bohner, D. O'Regan, and A. Peterson, "Dynamic equations on time scales: a survey," *Journal of Computational and Applied Mathematics*, vol. 141, no. 1-2, pp. 1–26, 2002.

[3] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäauser, Boston, Mass, USA, 2001.

[4] L. Erbe, T. S. Hassan, A. Peterson, and S. H. Saker, "Oscillation criteria for half-linear delay dynamic equations on time scales," *Nonlinear Dynamics and Systems Theory*, vol. 9, no. 1, pp. 51–68, 2009.

[5] L. Erbe, T. S. Hassan, A. Peterson, and S. H. Saker, "Oscillation criteria for sublinear half-linear delay dynamic equations on time scales," *International Journal of Difference Equations*, vol. 3, no. 2, pp. 227–245, 2008.

[6] T. S. Hassan, "Kamenev-type oscillation criteria for second order nonlinear dynamic equations on time scales," *Applied Mathematics and Computation*, vol. 217, no. 12, pp. 5285–5297, 2011.

[7] Q. Zhang, "Oscillation of second-order half-linear delay dynamic equations with damping on time scales," *Journal of Computational and Applied Mathematics*, vol. 235, no. 5, pp. 1180–1188, 2011.

[8] Q. Lin, B. Jia, and Q. Wang, "Forced oscillation of second-order half-linear dynamic equations on time scales," *Abstract and Applied Analysis*, vol. 2010, Article ID 294194, 10 pages, 2010.

[9] S. R. Grace, R. P. Agarwal, and S. Pinelas, "Comparison and oscillatory behavior for certain second order nonlinear dynamic equations," *Journal of Applied Mathematics and Computing*, vol. 35, no. 1-2, pp. 525–536, 2011.

[10] R. P. Agarwal, M. Bohner, and S. H. Saker, "Oscillation of second order delay dynamic equations," *The Canadian Applied Mathematics Quarterly*, vol. 13, no. 1, pp. 1–17, 2005.

[11] O. Došlý, "Qualitative theory of half-linear second order differential equations," *Mathematica Bohemica*, vol. 127, no. 2, pp. 181–195, 2002.

[12] L. Erbe, A. Peterson, and S. H. Saker, "Oscillation criteria for second-order nonlinear dynamic equations on time scales," *Journal of the London Mathematical Society*, vol. 67, no. 3, pp. 701–714, 2003.

[13] I. V. Kamenev, "Integral criterion for oscillation of linear differential equations of second order," *Matematicheskie Zametki*, vol. 23, pp. 249–251, 1978 (Russian).

[14] Y. Şahiner, "Oscillation of second-order delay dynamic equations on time scales," *Nonlinear Analysis—Theory Methods & Applications*, vol. 63, pp. 1073–1080, 2005.

[15] S. H. Saker, "Oscillation criteria of second-order half-linear dynamic equations on time scales," *Journal of Computational and Applied Mathematics*, vol. 177, no. 2, pp. 375–387, 2005.

[16] R. Mařík, "Riccati-type inequality and oscillation criteria for a half-linear PDE with damping," *Electronic Journal of Differential Equations*, vol. 2004, no. 11, pp. 1–17, 2004.

[17] A. Toraev, "On oscillation properties of elliptic equations," *Differential Equations*, vol. 47, no. 1, pp. 119–125, 2011.

[18] R. Mařík, "Integral averages and oscillation criteria for half-linear partial differential equation," *Applied Mathematics and Computation*, vol. 150, no. 1, pp. 69–87, 2004.

[19] R. Mařík, "Oscillation criteria for PDE with $p$-Laplacian via the Riccati technique via the Riccati technique," *Journal of Mathematical Analysis and Applications*, vol. 248, no. 1, pp. 290–308, 2000.

[20] O. Došlý, "Oscillation and spectral properties of a class of singular self-adjoint differential operators," *Mathematische Nachrichten*, vol. 188, pp. 49–68, 1997.

[21] A. Toraev and G. I. Garadzhaeva, "Kneser estimates for coefficients of elliptic equations," *Doklady Akademii Nauk SSSR*, vol. 295, no. 3, pp. 546–548, 1987.

[22] A. Toraev, "Oscillation and nonoscillation of the solutions of elliptic equations," *Differential Equations*, vol. 22, no. 8, pp. 1002–1010, 1986.

[23] A. Toraev, "Oscillation of elliptic operators and the structure of their spectrum," *Doklady Akademii Nauk SSSR*, vol. 279, no. 2, pp. 306–309, 1984.

[24] A. Toraev, "The oscillation of solutions of elliptic equations," *Doklady Akademii Nauk SSSR*, vol. 280, no. 2, pp. 300–303, 1985.

[25] A. Toraev, "Criteria for oscillation and nonoscillation for elliptic equations," *Differential Equations*, vol. 21, no. 1, pp. 104–113, 1985.

[26] A. Toraev, "The oscillatory and nonoscillatory behavior of solutions of elliptic-type higher order equations," *Doklady Akademii Nauk SSSR*, vol. 259, no. 6, pp. 1309–1311, 1981.

[27] R. P. Agarwal, D. O'Regan, and S. H. Saker, "Oscillation criteria for second-order nonlinear neutral delay dynamic equations," *Journal of Mathematical Analysis and Applications*, vol. 300, no. 1, pp. 203–217, 2004.

[28] S. H. Saker, "Oscillation of second-order nonlinear neutral delay dynamic equations on time scales," *Journal of Computational and Applied Mathematics*, vol. 187, no. 2, pp. 123–141, 2006.

[29] R. P. Agarwal, D. O'Regan, and S. H. Saker, "Oscillation criteria for nonlinear perturbed dynamic equations of second-order on time scales," *Journal of Applied Mathematics & Computing*, vol. 20, no. 1-2, pp. 133–147, 2006.

[30] S. H. Saker, R. P. Agarwal, and D. O'Regan, "Oscillation of second-order damped dynamic equations on time scales," *Journal of Mathematical Analysis and Applications*, vol. 330, no. 2, pp. 1317–1337, 2007.

[31] M. T. Şenel, "Oscillation theorems for dynamic equation on time scales," *Bulletin of Mathematical Analysis and Applications*, vol. 3, no. 4, pp. 101–105, 2011.

[32] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, UK, 2nd edition, 1988.

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# OSCILLATION THEOREMS FOR DYNAMIC EQUATION ON TIME SCALES

(COMMUNICATED BY HÜSEYIN BOR)

M. TAMER ŞENEL

Abstract. By using the generalized Riccati transformation and the inequality technique, we establish a oscillation criterion for certain non-linear second-order dynamic equations with damping on a time scale.

## 1. Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his PhD thesis in 1988 in order to unify continuous and discrete analysis (see [1]). Since Stefan Hilger formed the definition of derivatives and integrals on time scales, several authors have expounded on various aspects of the new theory, see the paper by Agarwal, Bohner, O'Regan, and Peterson [2], Samir H. Saker, et.al [3] and the references cited [5]-[10]. A book on the subject of time scales by Bohner and Peterson [4] summarizes and organizes much of time scale calculus.

## 2. Some preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers R. Since we are interested in the oscillatory of solutions near infinity, we assume that $sup\mathbb{T} = \infty$, and define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$ . We assume that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers R. The forward and the backward jump operators on any time scale $\mathbb{T}$ are defined by $\sigma(t) := inf\{s \in \mathbb{T} : s > t\}, \rho(t) := sup\{s \in \mathbb{T} : s < t\}$.
A point $t \in \mathbb{T}, t > inf\mathbb{T}$, is said to be left-dense if $\rho(t) = t$, right-dense if $t < sup\mathbb{T}$ and $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. The graininess function ? for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$.

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For a function $f : \mathbb{T} \to R$ the (delta) derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}$$

if f is continuous at t and t is right-scattered. A useful formula is

$$f^{\sigma} = f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

We will use the product rule and the quotient rule for the derivative of the product fg and the quotient f/g

$$(fg)^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta} = f g^{\Delta} + f^{\Delta} g^{\sigma}, \left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta} g - f g^{\Delta}}{g g^{\sigma}}.$$

The function $f : \mathbb{T} \to R$ is called rd-continuous if it is continuous in right-dense points and if the left-sided limits exist in left-dense points.
In this paper we shall study the oscillations of the following nonlinear second-order dynamic equations with damping

$$(r(t)\Psi(x(t))x^{\Delta}(t))^{\Delta} + p(t)(x^{\Delta}(t))^{\sigma} + q(t)(fox^{\sigma}(t)) = 0 \tag{2.1}$$

when $\Psi(x(t))$, $p(t), q(t)$ and $r(t)$ are positive rd-continuous functions.
We will use some of following assumptions:
$(H_1) f : R \to$ R is such that $f(u)/u \geq K > 0$, $uf(u) > 0$ for $u \neq 0$ and some $K > 0$,
$(H_2) 0 < c_1 \leq \Psi(v) \leq c_2$ for all $v$,
$(H_3) \int_{t_0}^{\infty} (\frac{1}{r(t)c} e_{\frac{-p(t)}{r(t)c}}(t, t_0)) \Delta t = \infty$ , some $c > 0$.

Our attention is restricted to those solutions of (2.1) which exist on some half-line $[t_x, \infty)$ and satisfy $sup\{|x(t)| : t > \mathbb{T}\} > 0$ for any $T \geq t_x$ . We assume the standing hypothesis that (2.1) does possess such solutions. A solution x(t) of (2.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

## 3. Main results

**Theorem 3.1.** *Assume that $(H_1) - (H_3)$ holds. Furthermore, assume that there exist a positive real rd-functions differentiable functions z(t) such that*

$$\limsup_{t \to \infty} \int_{t_0}^{t} \left[ K z(s) q(s) - \frac{c_2 r(s) A^2(s)}{4 z(s)} \right] \Delta s = \infty \tag{3.1}$$

*where*

$$A(t) = z^{\Delta}(t) - \frac{z(t) p(t)}{c_2 r^{\sigma}(t)},$$

*then every solution of (2.1) is oscillatory.*

*Proof.* Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (2.1). Without loss of generality, we may assume that $x(t) > 0$ for $t \geq t_1 > t_0$. We shall consider only this case, since in view of $(H_2)$, the proof of the case when $x(t)$ is eventually negative is similar. Now, we claim that $x^{\Delta}(t)$ has a fixed sign on the interval $[t_2, \infty)$ for some $t_2 \geq t_1$. From (2.1), since $q(t) > 0$ and $f(x(t)) > 0$, we have

$$(r(t)\Psi(x(t))x^{\Delta}(t))^{\Delta} + p(t) x^{\Delta^{\sigma}}(t) = -q(t) f(x^{\sigma}(t)) < 0,$$

i.e.,

$$(r(t)\Psi(x(t))x^{\Delta}(t))^{\Delta} + p(t)x^{\Delta^{\sigma}}(t) < 0.$$

By setting $y(t) = r(t)\Psi(x(t))x^{\Delta}(t)$ , we immediately see that by $(H_2)$,

$$y^{\Delta}(t) + \frac{p(t)y^{\sigma}(t)}{r^{\sigma}(t)c_2} < 0$$

, which implies that $\left(y(t)e_{\frac{p(t)}{c_2 r^{\sigma}(t)}}\right)^{\Delta} < 0$. Then $y(t)e_{\frac{p(t)}{c_2 r^{\sigma}(t)}}$ is decreasing and thus y(t) is eventually of one sing. Then $x^{\Delta}(t)$ has a fixed sing for all sufficiently large t and we have one of the following:
First, we consider $x^{\Delta}(t) \geq 0$ on $[t_2, \infty)$ for some $t_2 \geq t_1$. Then in view of (2.1) we have

$$x(t) > 0,\ x^{\Delta}(t) \geq 0,\ (r(t)\Psi(x(t))x^{\Delta}(t))^{\Delta} \leq 0,\ t \geq t_2. \tag{3.2}$$

Define the function $w(t)$ by Riccati substitution

$$w(t) := z(t)\frac{r(t)\Psi(x(t))x^{\Delta}(t)}{x(t)},\ t \geq t_2 \tag{3.3}$$

Then $w(t) > 0$ and satisfies

$$w^{\Delta}(t) = \left[\frac{z^{\Delta}(t)}{x(t)}\right]^{\Delta}(r(t)\Psi(x(t))x^{\Delta}(t))^{\sigma} + \frac{z(t)}{x(t)}(r(t)\Psi(x(t))x^{\Delta}(t))^{\Delta}$$

In view of (2.1) and (3.2) we see that

$$\begin{aligned} w^{\Delta}(t) = \quad & z^{\Delta}(t)\frac{(r(t)\Psi(x(t))x^{\Delta}(t))^{\sigma}}{x^{\sigma}(t)} - \frac{z(t)x^{\Delta}(t)}{x(t)x^{\sigma}(t)}(r(t)\Psi(x(t))x^{\Delta}(t))^{\sigma} \\ - \quad & z(t)p(t)\frac{(x^{\Delta}(t))^{\sigma}}{x(t)} - z(t)q(t)\frac{f(x^{\sigma}(t))}{x(t)} \end{aligned} \tag{3.4}$$

However from (3.2), $(H_1) - (H_2)$ and

$$r(t)\Psi(x(t))x^{\Delta}(t) \geq (r(t)\Psi(x(t))x^{\Delta}(t))^{\sigma},\ x^{\sigma}(t) \geq x(t),$$

we have

$$\begin{aligned} w^{\Delta}(t) \leq z^{\Delta}(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \frac{z(t)x^{\Delta}(t)}{(r(t)\Psi(x(t))x^{\Delta}(t))^{\sigma}}(w^{\sigma}(t))^2 \quad & - \quad p(t)z(t)\frac{(r(t)\Psi(x(t))x^{\Delta}(t))^{\sigma}}{r^{\sigma}(t)\Psi^{\sigma}(x(t))x^{\sigma}(t)} \\ & - \quad z(t)q(t)\frac{f(x^{\sigma}(t))}{x^{\sigma}(t)} \end{aligned} \tag{3.5}$$

$$w^{\Delta}(t) \leq -Kz(t)q(t) + A(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - z(t)\frac{((w^{\sigma}(t))^2}{c_2(z^{\sigma}(t))^2 r(t)} \tag{3.6}$$

where

$$A(t) = \left[z^{\Delta}(t) - \frac{z(t)p(t)}{c_2 r^{\sigma}(t)}\right].$$

Then

$$\begin{aligned} w^{\Delta}(t) &\leq -Kz(t)q(t) + \frac{c_2 r(t)A^2(t)}{4z(t)} - \left[\sqrt{\frac{z(t)}{c_2 r(t)}}\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \frac{A(t)}{2}\sqrt{\frac{c_2 r(t)}{z(t)}}\right]^2 \\ &\leq -\left[Kz(t)q(t) - \frac{c_2 r(t)A^2(t)}{4z(t)}\right] \end{aligned} \tag{3.7}$$

Integration from $t_3$ to t, we obtain

$$w(t) - w(t_3) \leq -\int_{t_3}^{t}\left[Kz(s)q(s) - \frac{c_2 r(s)A^2(s)}{4z(s)}\right]\Delta s \tag{3.8}$$

which yields

$$\int_{t_3}^{t}\left[Kz(s)q(s) - \frac{c_2 r(s)A^2(s)}{4z(s)}\right]\Delta s \leq w(t_3) - w(t) < w(t_3),$$

for all large t. This is contrary to (3.1).
Next, we consider $x^{\Delta}(t) < 0$ for $t > t_2 \geq t_1$ .
Define the function $u(t) = -r(t)\Psi(x(t))x^{\Delta}(t)$. The from (2.1)and $(H_2) - (H_3)$, we have

$$u^{\Delta}(t) + \frac{p(t)}{c_1 r(t)}u(t) \geq 0 \Rightarrow u(t) \geq u(t_2)e_{\frac{-p(t)}{c_1 r(t)}}(t, t_2),$$

Thus

$$x^{\Delta}(t) \leq -u(t_2)\left(\frac{1}{c_2 r(t)}e_{\frac{-p(t)}{c_1 r(t)}}(t, t_2)\right). \tag{3.9}$$

Integrating (3.9 ) from $t_2$ to t, we have

$$x(t) - x(t_2) \leq r(t_2)\Psi(x(t_2))x^{\Delta}(t_2)\int_{t_2}^{t}\left(\frac{1}{c_2 r(t)}e_{\frac{-p(t)}{c_1 r(t)}}(t, t_2)\right)\Delta s.$$

Condition $(H_3)$ implies that x(t) is eventually negative, which is a contradiction. The proof is complete. □

**Corollary 3.2.** *Assume that $(H_1) - (H_4)$ hold. If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[Kq(s) - \frac{r(s)p^2(s)}{4c_2(r^{\sigma}(s))^2}\right]\Delta s = \infty \tag{3.10}$$

*then every solution of (2.1) is oscillatory.*

**Corollary 3.3.** *Assume that $(H_1) - (H_3)$ hold. If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[Ksq(s) - \frac{c_2 r(s)}{4s}\left(1 - \frac{sp(s)}{c_2 r^{\sigma}(s)}\right)^2\right]\Delta s = \infty \tag{3.11}$$

*then every solution of (2.1) is oscillatory.*

**Example 2.4.** Consider the second order dynamic equation

$$\left(\frac{1}{t}(\frac{1}{4} + e^{-|x(t)|})x^{\Delta}(t)\right)^{\Delta} + \frac{1}{t^2}(x^{\Delta}(t))^{\sigma} + \frac{1}{t}x^{\sigma} = 0, t \geq 1. \tag{3.12}$$

The conditions $(H_1)-(H_3)$ are satisfied. First let consider Corollary 3.2. Assume that K=1, $c_2 = 5/4$ and $\mathbb{T} = 2^N = \{t : t = 2^k, k = N\}$. So we have $\sigma(t) = 2t$. It remains to show condition (3.10), we have

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[ Kq(s) - \frac{r(s)p^2(s)}{4c_2(r^\sigma(s))^2} \right] \Delta s = \limsup_{t\to\infty} \int_{1}^{t} \left[ \frac{1}{s} - \frac{4}{5s} \right] \Delta s = \infty.$$

Then, by Corollary 3.2, every solution of (3.12) oscillates. Now let consider Corollary 3.3. Assume that K=1, $c_2 = 5/4$ and $\mathbb{T} = \mathbb{Z}$. So we have $\sigma(t) = t$. From condition (3.11),

$$\begin{aligned} \limsup_{t\to\infty} \int_{t_0}^{t} \left[ Ksq(s) - \frac{c_2 r(s)}{4s} \left( 1 - \frac{sp(s)}{c_2 r^\sigma(s)} \right)^2 \right] \Delta s &= \limsup_{t\to\infty} \int_{1}^{t} \left[ 1 - \frac{1}{2s^2} \right] \Delta s \\ &= \sum_{t=1}^{\infty} \left[ 1 - \frac{1}{2t^2} \right] = \infty, \end{aligned}$$

then, by Corollary 3.3, every solution of (3.12) oscillates.

## References


[1] S. Hilger, *Analysis on measure chains?a unified approach to continuous and discrete calculus*, Results Math. **18** (1990) 18-56.

[2] R.P. Agarwal, M. Bohner, D. O'Regan, A. Peterson, *Dynamic equations on time scales: A survey, in: R.P. Agarwal, M. Bohner, D. O'Regan (Eds.), Special Issue on ?Dynamic Equations on Time Scales?* , J. Comput. Appl. Math. **141** (1?2) (2002) 1-26.

[3] Samir H. Saker, Ravi P. Agarwal, Donal O'Regan, *Oscillation of second-order damped dynamic equations on time scales*, J.Math.Anal. Appl., **330** (2007) 1317-1337.

[4] M. Bohner, A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.

[5] M. Bohner, L. Erbe, A. Peterson, *Oscillation for second order dynamic equations on time scale*, J. Math., Anal., Appl., **301** (2005) 491-507.

[6] M. Bohner, S.H. Saker, *Oscillation of second order nonlinear dynamic equations on time scales*, Rocky Mountain J. Math. **34** (2004) 1239-1254.

[7] E. Thandapani, B.S. Lalli, *Oscillation criteria for a second order damped difference equation*, Appl. Math. Lett. **8** (1995) 1-6.

[8] S.R. Grace, R.P. Agarwal, B. Kaymakçalan, W. Sae-jie, *Oscillation theorems for second order nonlinear dynamic equations*, J. Appl. Math. Comput. **32** (2010) 205-218.

[9] S.R. Grace, *Oscillation Theorems for nonlinear differential equations of second order*, J. of Math., Anal., App., **171** (1992) 220-241.

[10] B. Ayanlar, A. Tiryaki, *Oscillation theorems for nonlinear second order differential equations with damping*, Acta Math. Hungar **89** (2000) 1-13.



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# Oscillation theorems for second-order damped dynamic equation on time scales

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# Oscillation Theorems for Second-Order Damped Dynamic Equation on Time Scales

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**Abstract.** In this paper, we establish some new oscillation criteria for the second-order damped dynamic equation

$$(r(t)\Psi(x^{\Delta}(t))^{\Delta}+p(t)\Psi(x^{\Delta}(t))+q(t)f(x^{\sigma}(t))=0,\ t\in\mathbb{T},$$

where $\Psi(t)$, $f(t)$, $p(t), q(t)$ and $r(t)$ are rd-continuous functions. By using a generalized Riccati transformation and integral averaging technique, we establish some new sufficient conditions which ensure that every solution of this equation oscillates.

**Keywords:** Oscillation, Dynamic equations, Time scales
**PACS:** 02.30.Hq, 04.40.Dg

## INTRODUCTION

Much recent attention has been given to dynamic equations on time scales, or measure chains, and we refer the reader to the landmark paper of S. Hilger [1] for a comprehensive treatment of the subject. Since then, several authors have expounded on various aspects of this new theory; see the survey paper by Agarwal, Bohner, O'Regan and Peterson [2]. A book on the subject of time scales by Bohner and Peterson [3] also summarizes and organizes much of the time scale calculus. The various type oscillation and nonoscillation criteria for solutions of ordinary and dynamic equations have been studied extensively in a large cycle of works (see, [4-11] ). In [4], Saker, Agarwal, O'Regan considered the non-linear dynamic equation

$$(a(t)x^{\Delta}(t))^{\Delta}+p(t)x^{\Delta^{\sigma}}(t)+q(t)f(x^{\sigma}(t))=0,$$

where $a(t), p(t), q(t)$ are positive rd-continous functions. They gave some sufficient conditions for oscillation.

In this paper, we shall study the oscillations of the following nonlinear second-order dynamic equations with damping

$$(r(t)\Psi(x^{\Delta}(t))^{\Delta}+p(t)\Psi(x^{\Delta}(t))+q(t)f(x^{\sigma}(t))=0, \tag{1}$$

where $\Psi(t)$, $f(t)$, $p(t), q(t)$ and $r(t)$ are rd-continuous functions.

By using a generalized Riccati transformation and integral averaging technique, we establish some new sufficient conditions which ensure that every solution of this equation oscillates. Throughout this paper, we will assume the following hypotheses:
$(H_1)$ $p(t),\ q(t)\in C_{rd}(\mathbb{R},\mathbb{R}^{+})$,
$(H_2)$ $\Psi:\mathbb{T}\to\mathbb{R}$ is such that $\Psi^2(u)\le\kappa u\Psi(u)$ for $\kappa>0$, $u\neq 0$,
$(H_3)$ $f:\mathbb{R}\to\mathbb{R}$ is such that $\frac{f(u)}{u}\ge\lambda>0$, and $uf(u)>0,\ u\neq 0$,
$(H_4)$ $r(t)\in C^1_{rd}([t_0,\infty),\mathbb{R}^{+}),\ \int_{t_0}^{\infty}(\frac{1}{r(t)}e_{\frac{-p(t)}{r(t)}}(t,t_0))\Delta t=\infty$.

Our attention is restricted to those solutions of (1) which exist on some half-line $[t_x,\infty)$ and satisfy $sup\{|x(t)|:t>\mathbb{T}\}>0$ for any $T\ge t_x$ . We assume the standing hypothesis that (1) does possess such solutions. A solution x(t) of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

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## MAIN RESULTS

**Theorem 1.** Assume that $(H_1)-(H_4)$ hold. Furthermore, assume that there exists a positive real rd-functions differentiable functions $z(t)$ such that

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\lambda z(s)q(s)-\frac{\kappa r(s)A^2(s)}{4z(s)}\right]\Delta s=\infty, \tag{2}$$

where

$$A(t)=\left[z^{\Delta}(t)-\frac{z(t)p(t)}{r(t)}\right]. \tag{3}$$

Then every solution of (1) is oscillatory.
**Corollary 2.** Assume that $(H_1)-(H_4)$ hold. If

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\lambda q(s)-\frac{\kappa p^2(s)}{4r(s)}\right]\Delta s=\infty, \tag{4}$$

then every solution (1) is oscillatory.
**Corollary 3.** Assume that $(H_1)-(H_4)$ hold. If

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[s^{\gamma}\lambda q(s)-\frac{\kappa(r(s)(s^{\gamma})^{\Delta}-s^{\gamma}p(s))^2}{4r(s)}\kappa s^{-\gamma}\right]\Delta s=\infty, \tag{5}$$

then every solution (1) is oscillatory.
**Example.** Consider the dynamic equation

$$\left(\frac{1}{t^2}\left(\frac{x^{\Delta}(t)}{1+(x^{\Delta}(t))^2}\right)\right)^{\Delta}+\frac{1}{t^2}\left(\frac{x^{\Delta}(t)}{1+(x^{\Delta}(t))^2}\right)+\frac{1}{t}\left(\frac{1}{x^{\sigma}(t)}\right)=0,\ t>0,$$

where $r(t)=\frac{1}{t^2},\ p(t)=\frac{1}{t^2},\ q(t)=\frac{1}{t},\ \Psi(x^{\Delta})=\frac{x^{\Delta}(t)}{1+(x^{\Delta}(t))^2}$. All conditions of Corollary 3 are satisfied. Hence it is oscillatory.
**Corollary 4.** Assume that $(H_1)-(H_4)$ hold. If

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[R(s,t_0)\lambda q(s)-\frac{\kappa r(s)}{4R(s,t_0)}\left((R(s,t_0))^{\Delta}-\frac{R(s,t_0)p(s)}{r(s)}\right)^2\right]\Delta s=\infty,$$

where $R(t,t_0)=\int_{t_0}^{t}\frac{1}{r(s)}\Delta s$, then every solution (1) is oscillatory.
Now, let us introduce the class of functions $\mathbb{R}$ which will be extensively used in the sequel. Let $\mathbb{D}_0\equiv\{(t,s)\in\mathbb{T}^2: t>s\geq t_0\}$ and $\mathbb{D}\equiv\{(t,s)\in\mathbb{T}^2: t\geq s\geq t_0\}$. The function $H\in C_{rd}(\mathbb{D},\mathbb{R})$ is said belongs to the class $\mathfrak{R}$ if
(i)$H(t,t)=0,\ t\geq t_0,\ H(t,s)>0$, on $\mathbb{D}_0$,
(ii) H has a continuous $\Delta$-partial derivative $H_s^{\Delta}(t,s)$ on $\mathbb{D}_0$ with respect to the second variable.(H is rd-continuous function if H is rd-continuous function in t and s.)
**Theorem 5.** Assume that $(H_1)-(H_4)$ hold.Let z(t) be positive real rd-functions differentiable function and let $H:\mathbb{D}\to\mathbb{R}$ be rd-continuous function such that H belongs to the class $\mathfrak{R}$ and where

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\lambda z(s)q(s)-\frac{\kappa r(s)(\varphi(t,s))^2}{4z(s)H(t,s)}\right]\Delta s=\infty, \tag{6}$$

$$\varphi(t,s)=z^{\sigma}(s)H_s^{\Delta}(t,s)+H(t,s)A(s).$$

Then every solution of (1) is oscillatory.
**Corollary 6.** Suppose that the assumptions of Theorem 5 hold. If

$$\limsup_{t\to\infty}\frac{1}{H(t,t_2)}\int_{t_2}^{t}H(t,s)\left[\lambda q(s)-\frac{\kappa r(s)}{4z(s)}\left(\frac{H_s^{\Delta}(t,s)}{H(t,s)}-\frac{p(s)}{r(s)}\right)^2\right]\Delta s=\infty,$$



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then every solution of (1) is oscillatory.

**Corollary 7.** Let the assumption (6) in Theorem 5 be replaced by

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\lambda z(s)q(s)=\infty,$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[\frac{r(s)}{z(s)H(t,s)}\left(H(t,s)A(s)+z^{\sigma}(s)H_s^{\Delta}(t,s)\right)^2\right]\Delta s<\infty.$$

Then every solution of (1) is oscillatory.

Now, suppose that $p(t)=0$.

**Theorem 8.** Assume that $(H_1)-(H_4)$ hold. Furthermore, assume that there exists a positive real rd-continuous function $z(t)$ such that

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\lambda z(s)q(s)-\frac{\kappa r(s)}{4z(s)}(z^{\Delta}(s))^2\right]\Delta s=\infty, \tag{7}$$

then every solution of Eq. (1) is oscillatory.

**Corollary 9.** Assume that $(H_1)-(H_4)$ hold. If

$$\limsup_{t\to\infty}\int_{t_0}^{t}\lambda q(s)\Delta s=\infty, \tag{8}$$

then equation (1) is oscillatory.

**Theorem 10.** Assume that $(H_1)-(H_4)$ hold.Let $z(t)$ be positive real rd-functions differentiable function and let $H:\mathbb{D}\to\mathbb{R}$ be rd-continuous function such that H belongs to the class $\mathfrak{R}$ . If

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\lambda z(s)q(s)-\frac{\kappa r(s)C^2(t,s)}{4z(s)H(t,s)}\right]\Delta s=\infty,$$

where

$$C(t,s)=z^{\sigma}(s)H_s^{\Delta}(t,s)+H(t,s)z^{\Delta}(s),$$

then equation (1) is oscillatory.

**Corollary 11.** Assume that $(H_1)-(H_4)$ hold. Let $z(t)=1$. If

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left(\lambda H(t,s)q(s)-\kappa r(s)(H_s^{\Delta}(t,s))^2\right)\Delta s=\infty,$$

then every solution of (1) is oscillatory.

We assume that $f:\mathbb{R}\to\mathbb{R}$ is such that $f'(u)\geq k$ for $u\neq 0$ and some $k>0$.

**Theorem 12.** Assume that $(H_1)-(H_4)$ hold. Furthermore, assume that there exists a positive real rd-continuous function $z(t)$ such that

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[z(s)q(s)-\frac{\kappa r(s)A^2(s)}{4kz(s)}\right]\Delta s=\infty, \tag{9}$$

where $A(t)$ is as defined Theorem 1. Then every solution of (1) is oscillatory.

**Theorem 13.** Assume that $(H_1)-(H_4)$ hold. Let z(t) be positive real rd-functions differentiable function and let $H:\mathbb{D}\to\mathbb{R}$ be rd-continuous function such that H belongs to the class $\mathfrak{R}$ and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)z(s)q(s)-\frac{\kappa r(s)}{4kz(s)H(t,s)}B^2(t,s)\right]\Delta(s)=\infty, \tag{10}$$

$$B(t,s)=H(t,s)A(t)+H_s^{\Delta}(t,s)$$

and $A(t)$ is same as Theorem 1. Then every solution of (1) is oscillatory.



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## REFERENCES


1. S. Hilger, *Results Math.* **18**, 18–56 (1990).
2. R.P. Agarwal, M. Bohner, D. O'Regan, and A. Peterson, *J. Comput. Appl. Math.* **141(1-2)**, 1–26 (2002).
3. M. Bohner, and A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser, Boston, 2001.
4. S.H. Saker, R.P. Agarwal, and D. O'Regan, *J. Math.Anal. App.* **330**, 1317-1337, (2007).
5. M. Bohner, L. Erbe, and A. Peterson, *J. Math. Anal. App.* **301**, 491–507 (2005).
6. M. Bohner, and S.H. Saker, *Rocky Mountain J. Math.* **34**, 1239–1254 (2004).
7. T.S. Hassan, L. Erbe, and A. Peterson, *Com. Math. Appl.* **59**, 550–558 (2010).
8. R.P. Agarwal, D. O'Regan, S.H. Saker, *J. Math. Anal. Appl.* **300**, 203–217 (2004).
9. L. Erbe, *Canad. Appl. Math. Quart.* **9**, 1–31 (2001).
10. M. T. Şenel, *Bull. Math. Anal. Appl.* **3(4)**, 101–105, (2011).
11. M. T. Şenel, *Abst. App. Anal.* **2012**, Article ID 253107, 18 pages, doi:10.1155/2012/253107.




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## $Z_2Z_4$-additive Cyclic Codes

Irfan Siap[1] and Taher Abualrub

n this paper, we study $Z_2Z_4$ -additive cyclic codes. We will show that these codes will be identified as $Z_4[x]$-submodules of the ring $R_{r,s}=Z_2[x]/(x^r-1)\times Z_4[x]/(x^s-1)$. A set of generator polynomials for these odes as $Z_4[x]$-submodule of the ring Rr,s will be found. In addition, we obtain a $Z_2Z_4$-additive cyclic odes that leads to a new binary non-linear code that has parameters (33, 1687, 15).

**Keywords:** Additive Cyclic Codes, Modules, Mixed Codes.

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## On Some Codes over $F_p^k+...+u^m F_p^k$

Yasemin Cengellenmis[1]

We define a new map between the codes over $F_p^k+...+u^m F_p^k$ and F_p^k.The images of $(1-u^m)$-cyclic and cyclic codes over $F_p^k+...+u^mF_p^k$ under this map are obtained.In this way, the different characterization of the codes over $F_p^k$ is obtained.

**Keywords:** Constacyclic Codes, Cyclic Codes, Ouasi-cyclic Codes, The Gray map

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## Oscillation Theorems for Second-Order Nonlinear Dynamic Equation on Time Scales

M. Tamer Şenel[1], Onur Saldir[2] and Nadide Utku[3]

This paper concerns the oscillation of solutions to second order non-linear dynamic equation $(r(t)x^{\Delta}(t))^{\Delta} + p(t)f(x^{\sigma}(t))g(x^{\Delta}(t)) = 0$ on a time scale $\mathbb{T}$ which is unbounded above.

**Keywords**: Time scales, dynamic equation, oscillation.

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# ZAMAN SKALASINDA 3. MERTEBEDEN NÖTRAL DİNAMİK DENKLEMİN ÇÖZÜMLERİNİN DAVRANIŞI

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## ÖZET

Bu çalışmada zaman skalasında tanımlı 3. mertebeden nötral dinamik denklem

$$\left[ r(t)([x(t)+\int_a^b p(t,\mu)x[\tau(t,\mu)]\Delta\mu]^{\Delta\Delta})^{\gamma} \right]^{\Delta} + \int_c^d q(t,\xi)\, f(x[\phi(t,\xi)])\Delta\xi = 0, \quad t \in \mathrm{T} \qquad (1)$$

incelendi. Burada γ pozitif tek tamsayıların bir oranıdır ve r(t)>0 rd- sürekli fonksiyondur.

Genelleştirilmiş Riccati dönüşümü ve integral averaging tekniğini kullanarak (1) denkleminin çözümlerinin davranışı incelendi.

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**Anahtar Kelimeler**: 3. Mertebe, Nötral dinamik denklem, Zaman Skalası, Salınımlılık

## KAYNAKLAR

[1] A.R: Hammons, V. Kumar, A.R. Calderbank, N.J.A. Sloane, P. Sole, The $\mathbb{Z}_4$-Linearity of Kerdock, Preparata, Goethals and Related Codes, *IEEE Transactions on Information Theory* 40:301-319, 1994,

[2] Z. X. Wan, Quaternary Codes, *World Scientific*, 1997.

**Oscillation Theorems for Second-Order Damped Dynamic Equation on Time Scales**

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**Abstract**

Much recent attention has been given to dynamic equations on time scales, or measure chains, and we refer the reader to the landmark paper of S. Hilger [1] for a comprehensive treatment of the subject. A book on the subject of time scales by Bohner and Peterson [2] also summarizes and organizes much of the time scale calculus.

In this paper we shall study the oscillations of the following nonlinear second-order dynamic equations with damping

$$(r(t)\Psi(x^{\Delta}(t))^{\Delta} + p(t)\Psi(x^{\Delta}(t)) + q(t)f(x^{\sigma}(t)) = 0,\ t \in \mathbb{T}, \tag{1}$$

where $\Psi(t)$, $f(t)$, $p(t), q(t)$ and $r(t)$ are rd-continuous functions. By using a generalized Riccati transformation and integral averaging technique, we establish some new sufficient conditions which ensure that every solution of this equation oscillates. Throughout this paper, we will assume the following hypotheses:

$(H_1)$ $p(t),\ q(t) \in C_{rd}(\mathbb{R}, \mathbb{R}^+)$,

$(H_2)$ $\Psi : \mathbb{T} \to \mathbb{R}$ is such that $\Psi^2(u) \leq \kappa u \Psi(u)$ for $\kappa > 0$, $u \neq 0$,

$(H_3)$ $f : \mathbb{R} \to \mathbb{R}$ is such that $\frac{f(u)}{u} \geq \lambda > 0$, and $uf(u) > 0,\ u \neq 0$,

$(H_4)$ $r(t) \in C^1_{rd}([t_0, \infty), \mathbb{R}^+)$, $\int_{t_0}^{\infty} (\frac{1}{r(t)} e_{\frac{-p(t)}{r(t)}}(t, t_0))\Delta t = \infty$.

**References**

[1] S. Hilger, Analysis on measure chains A unified approach to continuous and discrete calculus, *Results Math.*, 18 , 18-56, 1990.

[2] M. Bohner, A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, *Birkhäuser*, Boston, 2001.

[3] Samir H. Saker, Ravi P. Agarwal, Donal O'Regan, Oscillation of second-order damped dynamic equations on time scales, J.Math.Anal. and App., 330, 1317-1337, 2007.

[4] Taher S. Hassan, Lynn Erbe, Allan Peterson, Oscillation Theorems of Second Order Superlinear Dynamic Equations with Damping on Time Scales, Com. Math. Appl., 59, 550-558, 2010.

[5] M. T. Şenel, Oscillation theorems for dynamic equation on time scales, *Bull. Math. Anal. Appl.*, 3, no.4, 101-105, 2011.

[6] M. T. Şenel, Kamenev-Type Oscillation Criteria for the Second-Order Nonlinear Dynamic Equations with Damping on Time Scales, Abstract and Applied Analysis, Vol. 2012, Article ID 253107, 18 pages, doi:10.1155/2012/253107.

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